Index theory for quasi-crystals I. Computation of the gap-label group

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INDEX THEORY FOR QUASI-CRYSTALS
I. COMPUTATION OF THE GAP-LABEL GROUP

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Abstract. In this paper, we give a complete solution to the gap labelling conjecture for quasi-crystals. The method adopted relies on the index theory for laminations, and the main tools are the Connes-Skandalis longitudinal K-theory index morphism together with the measured index formula.

Contents

Introduction

The study of quasi-periodic tillings have gained ground during the last years [1, 6, 8, 11, 27, 28, 30, 40, 41, 46]. The space of (equivalence classes) of such tillings are usually *pathological* and the classical topological methods cannot distinguish them from finite sets. In order to get more insight into interesting invariants of these structures, one is naturally led to the *non commutative geometry* point of view, based on the study of appropriate non commutative C^*-algebras. Much of the C^*-algebras associated with quasi-periodic tillings turn out to be closely related with C^*-algebras of dynamical systems or laminations. The non commutative tools developed in the study of foliated spaces [18, 19, 24, 47] thus represent a promising machinery for producing topological and geometrical invariants of tillings.

The present work is a first of a series of papers devoted to the study of quasi-periodic tillings from the non commutative geometry point of view. We have restricted ourselves to the case of quasi-crystals where the corresponding lamination is equivalent to a foliated bundle, associated with an action of the abelian free group, on a Cantor space. The famous gap-labelling conjecture is stated in this case, and involves the simplest geometric current: an invariant measure on the Cantor space and the trivial cocycle on the free abelian group. The main result of the present paper is a positive answer to the gap-labelling conjecture for quasi-crystals.

Let us now explain in more details the framework and the Bellissard conjecture. One considers a Schrödinger operator

$$H = -\frac{\hbar^2}{2m}\Delta + V\,,$$

where Δ is the Laplacian on $\mathbb{R}^p$ and V is a potential. The set of observables affiliated to this Schrödinger operator is a C^*- algebra which contains the C^*-algebra generated by the resolvent operators of H. If H describes a particle in a homogeneous media, the physical properties of this media do not depend upon the choice of an origin in $\mathbb{R}^p$ and the algebra of observables also contains the C^*-algebra $C^*(H)$ generated by the operators $T_a(H - z\,\mathrm{Id})^{-1}T_{-a}$, where T_a is the translation operator corresponding to $a \in \mathbb{R}^p$.

To the Schrödinger operator H, one assigns a compact space Ω_H equipped with a minimal action of $\mathbb{R}^p$ such that the crossed product C^*-algebra $C(\Omega_H) \rtimes \mathbb{R}^p$ contains $C^*(H)$, see [6]. This space Ω_H is defined as the strong closure in $\mathcal{B}(L^2(\mathbb{R}^p))$ of the space of operators $T_a(H - z\,\mathrm{Id})^{-1}T_{-a}$, where $a \in \mathbb{R}^p$, and z is a fixed complex number in the resolvent of H. The action of $\mathbb{R}^p$ on Ω_H is induced by translations and up to a $\mathbb{R}^p-$equivariant homeomorphism, the space Ω_H is independant on the choice of z in the resolvent of H.

A typical potential for motion of conduction electrons is given by $V(x) = \sum_{y\in L} v(x-y)$, where L is the point set of equilibrium positions of atoms and v is the effective potential for a valence electron near an atom (see [9]). In [9], J. Bellissard, D. Hermann and M. Zarrouati had attached to this point set L a geometric compact space Ω_L, called the hull of L and equipped with a minimal action of $\mathbb{R}^p$. Let ν_L be the uniform discrete measure on $\mathbb{R}^p$ supported by L, i.e. for every compactly supported continuous fonction f, $\nu_L(f) = \sum_{y\in L} f(y)$. Then Ω_L is by definition the weak-$*$ closure of the family of translations of ν_L by the elements of $\mathbb{R}^p$. Note that $V = \nu_L * v$ and more generally, for $\nu \in \Omega_L$ one can set

$$H_\nu = -\frac{\hbar^2}{2m}\Delta + \nu * v\,.$$

For every complex number z in the resolvent of the operator H, the map

$$\nu \in \Omega_L \longmapsto (H_\nu - z\,\mathrm{Id})^{-1} \in \Omega_H$$

is continuous, equivariant and surjective [9]. Thus, the crossed product C^*-algebra $C(\Omega_H) \rtimes \mathbb{R}^p$ imbeds in the crossed product C^*-algebra $C(\Omega_L) \rtimes \mathbb{R}^p$ and in particular $C(\Omega_L) \rtimes \mathbb{R}^p$ contains $C^*(H)$. The main advantage of dealing with Ω_L rather than Ω_H is that Ω_L only depends on the geometry of L. For instance, if L is given by a rank p lattice $\mathcal{R}$ in $\mathbb{R}^p$, then the hull of L is $\mathbb{R}^p/\mathcal{R}$. Actually in this case $C^*(H)$ can be computed by using Bloch theory [6] and one can check that $C^*(H) = C(B)\otimes\mathcal{K}$, where $\mathcal{K}$ is the elementary C^*-algebra of compact operators on a separable Hilbert space and B is the Brillouin zone, i.e. $B = \mathbb{R}^p/\mathcal{R}^*$ where $\mathcal{R}^*$ is the reciprocal lattice of $\mathcal{R}$.

On the other hand, one can also define the *integrated density of states* $E \mapsto \mathcal{N}(E)$, associated with the Schrödinger operator H, see [6]. Recall that $\mathcal{N}(E)$ is the number of states (per unit of volume) corresponding to eigenvalues less or equal to E. The remarkable result of [9] is that the values of the integrated density of states on gaps of the spectrum, are contained in a countable subgroup of $\mathbb{R}$ that only depends on L. If $\mathbb{P}$ is a $\mathbb{R}^p-$invariant ergodic probability measure on Ω_L, then $\mathbb{P}$ induces a trace $\tau^{\mathbb{P}}$ on the crossed product C^*-algebra $C(\Omega_L) \rtimes \mathbb{R}^p$, and this trace extends to the corresponding von-Neumann algebra. Let us denote by $\chi_{]-\infty,E]}$ the characteristic function of the set $]-\infty,E]$. In [6], J. Bellissard stated the so-called *Shubin formula* for H [55]:

$$\mathcal{N}(E) = \tau^{\mathbb{P}}(\chi_{]-\infty,E]}(H))\,.$$

So, if E belongs to a spectral gap of H, then $\chi_{]-\infty,E]}(H)$ is an idempotent which lives in the C^*-algebra $C(\Omega_L) \rtimes \mathbb{R}^p$ (recall that H is bounded below). In consequence and according to the Shubin formula, the values of $\mathcal{N}$ on spectral gaps of H are contained in the range of the additive map

$$\tau_*^{\mathbb{P}} : K_0(C(\Omega_L) \rtimes \mathbb{R}^p) \longrightarrow \mathbb{R}\,,$$

where $\tau_*^{\mathbb{P}}$ is the morphism induced in $K-$theory by the trace $\tau^{\mathbb{P}}$.

We now focus on the case where the point set L is a quasicrystal which is obtained by *the cut-and-project method* (see [27] and also [9]). With such a point set, J. Bellissard, E. Contensou and A. Legrand have associated in [11] (see also [9]) a useful discrete minimal dynamical system $(\mathbb{T}_L, \mathbb{Z}^p)$ which is Morita equivalent to the dynamical system $(\Omega_L, \mathbb{R}^p)$, and such that $\mathbb{T}_L$ is a Cantor set. Moreover, there is a canonical ergodic invariant probability measure μ on $\mathbb{T}_L$ such that

$$\tau_*^{\mathbb{P}}(K_0(C(\Omega_L) \rtimes \mathbb{R}^p)) = \tau_*^{\mu}(K_0(C(\mathbb{T}_L) \rtimes \mathbb{Z}^p)),$$

where τ^μ is the trace on $C(\mathbb{T}_L) \rtimes \mathbb{Z}^p$ induced by μ. Eventually, the image under $\tau_*^{\mathbb{P}}$ of the K-theory group $K_0(C(\Omega_L) \rtimes \mathbb{R}^p)$, and thus the gap-labelling group, is predicted by the following conjecture [6]:

Let Ω be a Cantor set equipped with an action of $\mathbb{Z}^p$ and with a $\mathbb{Z}^p$-invariant measure μ. The measure μ induces as above a trace τ^μ on the crossed product C^*-algebra $C(\Omega) \rtimes \mathbb{Z}^p$. Let us denote by $\mathbb{Z}[\mu]$ the additive subgroup of $\mathbb{R}$ generated by μ-measures of compact-open subsets of Ω. We make the assumption that Ω has no non-trivial compact-open invariant subset (this is clearly the case if the action of $\mathbb{Z}^p$ is minimal, which was the assumption in the original conjecture).

Conjecture 1 (The Bellissard gap-labelling conjecture) *Under the above assumptions, we have:*

$$\tau_*^{\mu}(K_0(C(\Omega) \rtimes \mathbb{Z}^p)) = \mathbb{Z}[\mu].$$

Many results in low dimensions have been obtained these last years [11, 10, 23], they all confirm the conjecture. The goal of the present paper is to give a general and uniform proof of the Bellissard conjecture for all dimensions. The method that we have adopted is new and is based on the index theory for laminations [20, 47] and on the properties of cyclic homology. It opens up the way to treat more complicated cocycles and these will be handled in forthcoming papers.

The proof goes more precisely as follows. We use the Baum-Connes isomorphism associated with the free abelian group to reduce the problem to the computation of the image under the trace, of indices of twisted Dirac operators along the leaves of a mapping-torus associated with the quasi-crystal. We then apply the Connes' measured index theorem for foliated spaces, as stated in [47]. The final argument is the integrality of the top-dimensional component of the longitudinal Chern character. In the process, we also obtain many spectral sequence results concerning periodic cyclic homology and longitudinal cohomology, and also the naturality of the longitudinal Chern character. As pointed out to us by J. Bellissard, the method used in the present paper works equally and with only minor modifications, for more complicated aperiodic solids.

The last step of the proof, namely the integrality of the top-dimensional component of the Chern character relies on a theorem of Forrest-Hunton [30] stating that the range of this Chern character is homogeneous. The referee of the present paper pointed out that there might be a mistake in the proof of this homogeneity in [30]. We are currently working on a new proof of the integrality theorem based on transverse index theory.

It is worthpointing out that two different proofs of the above conjecture have been recently and independently obtained in [36] and [7]. The first one is very similar to our method but rather uses a transfer technique, while the second one uses branched oriented flat manifolds.

The contents of this paper are more precisely as follows. In the first section, we recall the gap-labelling conjecture and explain how the Baum-Connes map reduces the problem to the index theorem. In the second section, we define the longitudinal cohomology and compute its top-dimensional component. In the third section, we compute the periodic cyclic (co)homology of the quasi-crystal and study an independently interesting longitudinal periodic cyclic homology which is used in the sequel. The longitudinal Chern character is also defined and studied at the end of this section. The last section is devoted to the proof of the conjecture.

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1. The K-theory approach to quasi-crystals

In this section, we shall state the Bellissard conjecture for quasi-crystals. We also define the longitudinal analytic index as a K-theory class of the C^*-algebra of the quasi-crystal and compute this class for longitudinal twisted Dirac operators in Kasparov's KK-theory. In particular, we explicitely prove that it induces the Baum-Connes isomorphism.

1.1. **Preliminaries on quasi-crystals.** Let Ω be a totally disconnected compact space. Assume that the group $\mathbb{Z}^p$ acts on Ω by homeomorphisms and that there exists a $\mathbb{Z}^p$-invariant positive measure on Ω. We fix such invariant probability measure and denote it by μ. The homeomorphism of Ω corresponding to the integer $n \in \mathbb{Z}^p$ will be denoted by n for short.

The action of $\mathbb{Z}^p$ on Ω induces an action of $\mathbb{Z}^p$ on the C^*-algebra $C(\Omega)$ of continuous complex valued functions on Ω. Thus we can define the crossed product C^*-algebra $C(\Omega) \rtimes \mathbb{Z}^p$ of $C(\Omega)$ by $\mathbb{Z}^p$ [52]. The measure μ induces a trace τ^μ on the C^*-algebra $C(\Omega) \rtimes \mathbb{Z}^p$ which, for a finite sum $f = \sum_{n\in\mathbb{Z}^p} f_n U_n \in C(\Omega) \rtimes \mathbb{Z}^p$, is given by the following formula:

$$\tau^\mu(\sum_{n\in\mathbb{Z}^p} f_n U_n) = \mu(f_0).$$

We thus obtain a group homomorphism $\tau^\mu_* : K_0(C(\Omega) \rtimes \mathbb{Z}^p) \to \mathbb{R}$ by setting:

$$\tau^\mu_*([e] - [e']) := \sum_i \tau^\mu(e_{ii}) - \sum_i \tau^\mu(e'_{ii})$$

where e and e' are self-adjoint idempotents in $M_\infty(C(\Omega) \rtimes \mathbb{Z}^p)$. Here $K_0(C(\Omega) \rtimes \mathbb{Z}^p)$ denotes the K-theory group of the C^*-algebra $C(\Omega) \rtimes \mathbb{Z}^p$.

In what follows, we shall denote by $C(\Omega, \mathbb{Z})$ the ring of continuous integer valued functions on Ω and by $\mathbb{Z}[\mu]$ the additive subgroup of $\mathbb{R}$ generated by μ-measures of compact-open subsets of Ω. Note that we also have:

$$\mathbb{Z}[\mu] = \left\{ \int_\Omega h \, d\mu, \quad h \in C(\Omega, \mathbb{Z}) \} \right\}.$$

It is easy to see that $K_0(C(\Omega)) \simeq C(\Omega, \mathbb{Z})$. We shall denote by $C(\Omega, \mathbb{Z})_{\mathbb{Z}^p}$ the coinvariants of the action of $\mathbb{Z}^p$ on $C(\Omega, \mathbb{Z})$, i.e. the quotient of $C(\Omega, \mathbb{Z})$ by the subgroup generated by elements of the form $n(f) - f$, where $f \in C(\Omega, \mathbb{Z})$ and $n \in \mathbb{Z}^p$.

For simplicity, we shall assume in the whole paper that the only $\mathbb{Z}^p$-invariant functions of $C(\Omega, \mathbb{Z})$ are the constant functions. This is true for example when the action is minimal. By a remarkable result of J. Bellissard and his collaborators [6, 9], the gap-labelling problem for a quasi-crystal is reduced to the computation of the image of the K-theory group $K_0(C(\Omega) \rtimes \mathbb{Z}^p)$ under the additive map τ^μ_* induced by the trace τ^μ, a purely mathematical question.

In the case $p = 1$, J. Bellissard proved in [6] (see also [23]), using the Pimsner-Voiculescu six term exact sequence, that the inclusion $C(\Omega) \hookrightarrow C(\Omega) \rtimes \mathbb{Z}$ induces an isomorphism $C(\Omega, \mathbb{Z})_{\mathbb{Z}} \simeq K_0(C(\Omega) \rtimes \mathbb{Z})$ and thus that:

$$\tau^\mu_*(K_0(C(\Omega) \rtimes \mathbb{Z})) = \mathbb{Z}[\mu]\,.$$

For $p = 2$, the computation of $K_0(C(\Omega) \rtimes \mathbb{Z}^2)$ was carried out by A. van Elst in [58] (see also [11]) using the Pimsner-Voiculescu exact sequence twice , The result is:

$$K_0(C(\Omega) \rtimes \mathbb{Z}^2) \simeq C(\Omega, \mathbb{Z})_{\mathbb{Z}^2} \bigoplus \mathbb{Z}.$$

More precisely, in this case:

- The inclusion $C(\Omega, \mathbb{Z})_{\mathbb{Z}^2} \hookrightarrow K_0(C(\Omega) \rtimes \mathbb{Z}^2)$ is induced by $C(\Omega) \hookrightarrow C(\Omega) \rtimes \mathbb{Z}^2$;
- The inclusion $\mathbb{Z} \hookrightarrow K_0(C(\Omega) \rtimes \mathbb{Z}^2)$ maps the canonical generator of $\mathbb{Z}$ to the image, under the morphism induced by the inclusion $C^*_r(\mathbb{Z}^2) \to C(\Omega) \rtimes \mathbb{Z}^2$, of the Bott generator in the K-theory of the C^*-algebra $C^*_r(\mathbb{Z}^2) \simeq C(\mathbb{T}^2)$.

Recall that the Bott generator is the unique traceless element of $K_0(C(\mathbb{T}^2))$ with Chern character equal to the normalized volume form of $\mathbb{T}^2$. Therefore the above computation gives again:

$$\tau_*^\mu(K_0(C(\Omega)\rtimes\mathbb{Z}^2))=\mathbb{Z}[\mu].$$

In the case $p=3$, the computation of $\tau_*^\mu(K_0(C(\Omega)\rtimes\mathbb{Z}^3))$ was recently performed in [10] and gave the same result. These computations lead J. Bellissard to state the following conjecture (see [10] and [9]):

Conjecture 1. *For every $p\geq 1$, we have:*

$$\tau_*^\mu(K_0(C(\Omega)\rtimes\mathbb{Z}^p))=\mathbb{Z}[\mu].$$

A positive answer to this conjecture provides a complete description of the gaps in the spectrum of the Schrödinger operator associated with the quasi-crystal, see [6]. Let us point out immediately that one of the two inclusions is quasi-trivial.

Lemma 1. $\mathbb{Z}[\mu]\subset\tau_*^\mu(K_0(C(\Omega)\rtimes\mathbb{Z}^p))$.

Proof. If Θ is any compact-open subset of Ω, we denote by χ_Θ the characteristic function of Θ, and by $[\chi_\Theta]$ the class of χ_Θ in $K_0(C(\Omega))$. Then the trace of the image of the element $[\chi_\Theta]$ under the map $K_0(C(\Omega))\to K_0(C(\Omega)\rtimes\mathbb{Z}^p)$ is equal to the measure of Θ. Now observe simply that $K_0(C(\Omega))$ is generated by the classes of characteristic functions of compact-open subsets of Ω. □

Lemma 1 shows that the Bellissard conjecture actually reduces to the inclusion $\tau_*^\mu(K_0(C(\Omega)\rtimes\mathbb{Z}^p))\subset\mathbb{Z}[\mu]$. As we have recalled above, the Bellissard conjecture has been verified for small p. Let us observe now a simple fact that will simplify our proof.

Lemma 2. *If the Bellissard conjecture is true for every $p\in 2\mathbb{N}^*$, then it is automatically true for every $p\geq 1$.*

Proof. We can embed the crossed product C^*-algebra $C(\Omega)\rtimes\mathbb{Z}^p$ in the crossed product C^*-algebra $C(\Omega)\rtimes\mathbb{Z}^{p+1}$ where $\mathbb{Z}^{p+1}$ acts on Ω through the projection $\mathbb{Z}^{p+1}\to\mathbb{Z}^p$ corresponding to the p first variables. Hence, the measure μ is also invariant under the action of $\mathbb{Z}^{p+1}$ and induces a trace on $C(\Omega)\rtimes\mathbb{Z}^{p+1}$, we also denote this trace by τ^μ. The injection $C(\Omega)\rtimes\mathbb{Z}^p\hookrightarrow C(\Omega)\rtimes\mathbb{Z}^{p+1}$ agrees with the traces and therefore, the following diagram is commutative:

$$\begin{tikzcd}
K_0(C(\Omega)\rtimes\mathbb{Z}^p) \arrow[rr] \arrow[dr, "\tau_*^\mu"'] & & K_0(C(\Omega)\rtimes\mathbb{Z}^{p+1}) \arrow[dl, "\tau_*^\mu"] \\
& \mathbb{R} &
\end{tikzcd}$$

□

We shall therefore assume when necessary that p is even.

1.2. **The longitudinal analytic index.** We gather in this subsection some basic definitions and properties about the index theory for the foliated mapping torus V_Ω associated with the quasi-crystal. Good references of the general material used in this section are, among others, [19, 20, 21, 22, 54].

As before, the space Ω is a Cantor set, and it is equipped with an action of $\mathbb{Z}^p$ by homeomorphisms. The mapping torus of the action of $\mathbb{Z}^p$ on Ω is the total space of a flat bundle. It is more precisely the quotient space:

$$V_\Omega=\Omega\times_{\mathbb{Z}^p}\mathbb{R}^p:=(\Omega\times\mathbb{R}^p)/\mathbb{Z}^p,$$

where $\mathbb{Z}^p$ acts diagonally on $\Omega\times\mathbb{R}^p$ and the action of $\mathbb{Z}^p$ on $\mathbb{R}^p$ is the usual one by translations. Recall that we have assumed that Ω has no non-trivial invariant compact-open subset.

Lemma 3. *The mapping torus V_Ω is connected.*

Proof. Let f be an integer valued continuous function on V_Ω. Let us denote by $\pi : \Omega \times \mathbb{R}^p \to V_\Omega$ the projection. Then the composition $g = f \circ \pi : \Omega \times \mathbb{R}^p \to \mathbb{Z}$ is a $\mathbb{Z}^p$-invariant integer valued function on $\Omega \times \mathbb{R}^p$. Therefore for any $\omega \in \Omega$, the function $g_\omega : \mathbb{R}^p \to \mathbb{Z}$ defined by $g_\omega(x) = g(\omega, x)$ is also continuous. But since $\mathbb{R}^p$ is connected this implies that g_ω does not depend on x and is a constant function. Hence the function g only depends on the Ω-component and is $\mathbb{Z}^p$-invariant. This shows by hypothesis that g must be constant and finally that f is constant. □

Denote by $\tilde{\mathcal{G}}$ the groupoid $\Omega \times \mathbb{R}^p \times \mathbb{R}^p$, which is the continuous family, indexed in Ω of the product groupoids $\{\omega\} \times \mathbb{R}^p \times \mathbb{R}^p$, $\omega \in \Omega$. The group $\mathbb{Z}^p$ acts diagonally, freely by groupoid homeomorphisms on $\tilde{\mathcal{G}}$, and the quotient groupoid will be denoted by $\mathcal{G}$. Notice that this groupoid has unit space V_Ω and that if $x = [\omega, m] \in V_\Omega$ then the set $\mathcal{G}^x$ of those elements of $\mathcal{G}$ which end at x is a smooth manifold and it is diffeomorphic to $\mathbb{R}^p$.

Denote by $\mathbb{T}^p$ the flat p-torus $\mathbb{T}^p := \mathbb{R}^p/\mathbb{Z}^p$. Any fibre of $\pi : V_\Omega \to \mathbb{T}^p$ is a transversal to the flat foliation which in addition cuts all the leaves. The restriction of the groupoid $\mathcal{G}$ to such a transversal is then the crossed product groupoid $\Omega \rtimes \mathbb{Z}^p$. Hence the C^*-algebra of the groupoid $\mathcal{G}$ can be identified with the C^*-algebra $[C(\Omega) \rtimes \mathbb{Z}^p] \otimes \mathcal{K}(H)$, where $C(\Omega) \rtimes \mathbb{Z}^p$ is the reduced crossed product C^*-algebra and $\mathcal{K}(H)$ is the elementary C^*-algebra of compact operators in a separable Hilbert space H [33].

To capture K-theory invariants of the transverse structure of the foliated space V_Ω, one has to compute the K-theory of the C^*-algebra $C(\Omega) \rtimes \mathbb{Z}^p$. One way to construct elements of this K-theory group is to use longitudinal elliptic operators and longitudinal index theory on the foliated space V_Ω.

Let E be a continuous longitudinally smooth vector bundle over V_Ω. We shall denote by $\tilde{E}$ the pull back of E to $\Omega \times \mathbb{R}^p$ equipped with the obvious action of $\mathbb{Z}^p$. The restriction of $\tilde{E}$ to each $\{\omega\} \times \mathbb{R}^p$ allows to define the space $\psi^\infty(\Omega \times \mathbb{R}^p, \tilde{E})$ of continuous families $(T_\omega)_{\omega\in\Omega}$ of classical pseudodifferential operator on $\mathbb{R}^p$ with coefficients in the vector bundle $\tilde{E}$. The algebra $\psi^\infty(\Omega \times \mathbb{R}^p, \tilde{E})$ fibers continuously over Ω with typical fibre the space $\psi^\infty(\mathbb{R}^p, \mathbb{C}^N)$ (where $\psi^\infty(\Omega \times \mathbb{R}^p, \tilde{E})$ is endowed with the topology of kernels, see [4, 12, 50]). Note that the group $\mathbb{Z}^p$ acts by automorphisms on $\psi^\infty(\Omega \times \mathbb{R}^p, E)$ and we get in this way a $\mathbb{Z}^p$-equivariant fibration.

Definition 1. (1) A pseudodifferential operator P of order m on the foliated bundle V_Ω, with coefficients in the longitudinally smooth continuous vector bundle E is a continuous section

$$P : \Omega \longrightarrow \psi^m(\Omega \times \mathbb{R}^p, \tilde{E}),$$

which is $\mathbb{Z}^p$-equivariant.

(2) The principal symbol of a pseudodifferential operator P is then defined as the family $\sigma(P) = (\sigma_\omega(P))_{\omega\in\Omega}$ such that $\sigma_\omega(P)$ is the principal symbol of the operator P_ω on $\mathbb{R}^p$. Hence $\sigma(P)$ is a continuous $\mathbb{Z}^p$-equivariant section

$$\sigma(P) \in C^{\infty,0}_m(\Omega \times (T^*\mathbb{R}^p \setminus \mathbb{R}^p), \mathrm{End}(\pi^*\tilde{E})),$$

where $\pi : \Omega \times (T^*\mathbb{R}^p \setminus \mathbb{R}^p) \to \Omega \times \mathbb{R}^p$ is the projection and $C^{\infty,0}_m$ means continuous smooth in the $\mathbb{R}^p$-direction sections which are positively m-homogeneous, i.e. such that:

$$\sigma(\omega, x, \lambda\xi) = \lambda^m \sigma(\omega, x, \xi), \quad \forall\lambda > 0.$$

A pseudodifferential operator P of order m is elliptic if for any $\omega \in \Omega$, $\sigma(P)(\omega, x, \xi)$ is an automorphism of the vector space $E_{\omega,x}$. This means that for any $\omega \in \Omega$, P_ω is elliptic. We point out that with P there is an associated family of Schwartz kernels indexed by Ω. Since this family is invariant under the action of $\mathbb{Z}^p$, we can construct an operator P^0 on the quotient space V_Ω with coefficients in the vector bundle E. Locally the total symbol of P^0 only depends on the covectors tangent to the leaves and is what is usually called a longitudinal pseudodifferential operator [20]. It is (longitudinally) elliptic if P is elliptic.

We shall only consider here longitudinal pseudodifferential operators on V_Ω with $\mathbb{Z}^p-$compactly supported distributional kernel and we denote by $\psi^m(V_\Omega|\Omega; E)$ the space of such operators. We set:

$$\psi^\infty(V_\Omega|\Omega; E) = \bigcup_{m\in\mathbb{Z}} \psi^m(V_\Omega|\Omega; E) \text{ and } \psi^{-\infty}(V_\Omega|\Omega; E) = \bigcap_{m\in\mathbb{Z}} \psi^m(V_\Omega|\Omega; E).$$

The Schwartz theorem enables to identify $\psi^{-\infty}(V_\Omega|\Omega; E)$ with the ideal $C_c^{\infty,0}(\mathcal{G}, \mathrm{End}(E))$ of longitudinally smooth continuous sections of the bundle $\mathrm{End}(E) = \mathrm{Hom}(r^*E, s^*E)$ over $\mathcal{G}$ [50]. Thus we have a short exact sequence of algebras

$$0 \to C_c^{\infty,0}(\mathcal{G}, \mathrm{End}(E)) \hookrightarrow \psi^\infty(V_\Omega|\Omega; E) \longrightarrow \mathcal{A}(V_\Omega|\Omega; E) \to 0, \tag{1}$$

where the quotient algebra $\mathcal{A}(V_\Omega|\Omega; E)$ is the algebra of longitudinal complete symbols on the foliated space V_Ω. This is a filtered algebra, the filtration being given by the order of the pseudodifferential operators. The symbol map induces an isomorphism between the $\mathbb{Z}$-graded algebra of the filtered algebra $\mathcal{A}(V_\Omega|\Omega; E)$ and the $\mathbb{Z}$-graded algebra of $\mathbb{Z}^p$-equivariant homogeneous sections of $\mathrm{End}(\tilde{E})$ over $\Omega \times (T^*\mathbb{R}^p \setminus \mathbb{R}^p)$. The classical parametrix theorem is true in the foliation case and we have [47][Proposition 7.12]:

Proposition 1. *If $P \in \psi^\infty(V_\Omega|\Omega; E)$ is elliptic then its class in $\mathcal{A}(V_\Omega|\Omega; E)$ is invertible.*

In other words, compactly supported elliptic pseudodifferential operators admit parametrices modulo the smooth algebra $C_c^{\infty,0}(\mathcal{G}, \mathrm{End}(E))$. Any elliptic pseudodifferential operator as in Definition 1 thus defines an index class obtained using the connecting map:

$$\partial : K_1(\mathcal{A}(V_\Omega|\Omega; E)) \longrightarrow K_0(C_c^{\infty,0}(\mathcal{G}, \mathrm{End}(E))),$$

associated with the short exact sequence (1). More precisely:

Definition 2. Let as before E be a vector bundle over V_Ω. Let P be a uniformly supported elliptic longitudinal pseudodifferential operator on V_Ω with coefficients in E. Then the image of the class $[\sigma(P)]$ of the principal symbol of P under the connecting map ∂ is called the analytic index of P and denoted by $\mathrm{Ind}_{V_\Omega}(P)$.

The above algebraic index class can also be defined using the language of quasi-isomorphisms [19] and is useful in view of using higher cyclic cohomology techniques [15]. In the sequel, we shall need the C^*-algebraic index as defined in [20]. It is worthpointing out that the extension map together with Morita equivalence, allows to get rid of the bundles and to end up with an index class in the $K-$theory of Connes C^*-algebra. This latter then coincides with the index class given in [20], see also [19]. Recall that Connes C^*-algebra of the foliation is the reduced C^*-algebra of the holonomy groupoid [20]. Again using Morita equivalence asociated with a complete transversal, the analytic index can also be viewed as an element of $K_0(C(\Omega) \rtimes \mathbb{Z}^p)$, where $C(\Omega) \rtimes \mathbb{Z}^p$ is the crossed product C^*-algebra. We shall also denote this latter (when no confusion can occur) by $\mathrm{Ind}_{V_\Omega}(P)$. Finally, we shall also need a $KK-$description of the analytic index and we proceed now to give the corresponding representative [22, 38].

Fix a continuous hermitian structure on the vector bundle E, which is longitudinally smooth. The space $C_c^{\infty,0}(\mathcal{G}, r^*E)$ of compactly supported continuous longitudinally smooth sections of r^*E over $\mathcal{G}$ is then endowed with a natural structure of a prehilbertian module over the regular convolution algebra $C_c^{\infty,0}(\mathcal{G})$ [22]. More precisely, given $\xi, \xi' \in C_c^{\infty,0}(\mathcal{G}, r^*E)$ and $k \in C_c^{\infty,0}(\mathcal{G})$, we set:

$$(\xi k)[\omega, m, m'] := \int_{\mathbb{R}^p} \xi[\omega, m, n] k[\omega, n, m'] dn \text{ and } \langle \xi, \xi' \rangle [\omega, m, m'] := \int_{\mathbb{R}^p} \langle \xi[\omega, n, m], \xi'[\omega, n, m'] \rangle dn.$$

The completion of $C_c^{\infty,0}(\mathcal{G}, r^*E)$ with respect to the above $C^*(\mathcal{G})$-valued inner product (where $C^*(\mathcal{G})$ is the reduced C^*-algebra of the groupoid $\mathcal{G}$) is then a Hilbert module over $C^*(\mathcal{G})$ that we denote by ϵ_E. In terms of continuous fields of Hilbert spaces over the space of leaves as described in [20], ϵ_E is the Hilbert module associated with the field of Hilbert spaces $H_{[\omega,m]} := L^2(\mathcal{G}^{[\omega,m]}; r^*E)$ where

$$\mathcal{G}^{[\omega,m]} = \{[\omega, m, m'], m' \in \mathbb{R}^p\} \simeq \mathbb{R}^p.$$

An elliptic pseudodifferential operator P then gives rise to a well defined operator on the Hilbert module ϵ_E that we still denote by P for simplicity, see [22]. This longitudinal operator corresponds, when the order k of P is zero and for any $[m, \omega] \in V_\Omega$ to the operator P_ω acting in a copy of $\mathbb{R}^p$ since $\mathcal{G}^{[m,\omega]} \cong \mathbb{R}^p$. In general one needs first to multiply P by a self-adjoint invertible longitudinal pseudodifferential operator of order $-k$, using for instance the $\mathbb{Z}^p$-invariant Laplace operator on $\mathbb{R}^p$ [3].

Now using the trivial pointwise representation ρ of the C^*-algebra $C(V_\Omega)$ of continuous complex valued functions on V_Ω, we obtain a triple (ϵ_E, ρ, P) which turns out to be a Kasparov triple over the pair of C^*-algebras $(C(V_\Omega), C^*(\mathcal{G}))$. Finally, we get a class $[P] := [(\epsilon_E, \rho, P)]$ in the Kasparov group $KK(C(V_\Omega), C^*(\mathcal{G}))$. Since V_Ω is compact, the projection $p : V_\Omega \to \{pt\}$ induces an element $[p] \in KK(\mathbb{C}, C(V_\Omega))$. Now the analytic index of P can be reinterpreted in Kasparov's theory as the class in $KK(\mathbb{C}, C^*(\mathcal{G}))$ given by:

$$\mathrm{Ind}_{V_\Omega}(P) = [p] \otimes_{C(V_\Omega)} [P],$$

where $\otimes_{C(V_\Omega)}$ is Kasparov product over the C^*-algebra $C(V_\Omega)$.

1.3. ***K*-theory of the crossed product.** The index map Ind_{V_Ω} furnishes a complete computation of the K-theory group of the reduced crossed product C^*-algebra $C(\Omega) \rtimes \mathbb{Z}^p$. We prove below that the analytic index map enables to construct an isomorphism:

$$\mu_\Omega^{\mathbb{Z}^p} : K_0(C(V_\Omega)) \simeq K_0(C(\Omega) \rtimes \mathbb{Z}^p).$$

Recall that $p = 2r$ is even. The map $\mu_\Omega^{\mathbb{Z}^p}$ is roughly speaking integration along the leaves in K-theory. Let $\partial : L^2(\mathbb{R}^p, S^+) \to L^2(\mathbb{R}^p, S^-)$ be the Dirac operator on $\mathbb{R}^p$, where S^+ and S^- are the two fundamental spin (trivial) bundles over $\mathbb{R}^p$. Note that as complex vector bundles, S^+ and S^- are isomorphic. Then ∂ is $\mathbb{Z}^p$-invariant and defines by classical arguments an equivariant K-homology class [38]:

$$[\partial] \in KK^{\mathbb{Z}^p}(C_0(\mathbb{R}^p), \mathbb{C}).$$

This operator yields a well defined longitudinal operator on V_Ω which is elliptic along the leaves and that we call the longitudinal Dirac operator. It does not depend on the Ω-component.

Let now e be a projection in $M_n(C(V_\Omega))$ and let E be the associated continuous hermitian vector bundle over V_Ω. Let $\tilde{e}$ be the $\mathbb{Z}^p-$invariant projection in $C(\mathbb{R}^p \times \Omega, M_n(\mathbb{C}))$ defined by e. We can assume that E is differentiable in the $\mathbb{R}^p$ direction, or equivalently that the projection $\tilde{e}$ is smooth in the $\mathbb{R}^p-$variable. The longitudinal Dirac operator ∂_Ω^e with coefficients in E is a longitudinal differential operator of order 1, which is elliptic along the leaves of the foliated space V_Ω. Recall that this operator is given by the continuous family $(\partial_\Omega^e)_\omega$:

$$(\partial_\Omega^e)_\omega := \tilde{e}_\omega(\partial \otimes id_{\mathbb{C}^n})\tilde{e}_\omega \;\; : \tilde{e}_\omega(L^2(\mathbb{R}^p, S^+ \otimes \mathbb{C}^n)) \longrightarrow \tilde{e}_\omega(L^2(\mathbb{R}^p, S^- \otimes \mathbb{C}^n)),$$

where ω runs over Ω and $\tilde{e}_\omega$ is the idempotent $x \to \tilde{e}(w, x)$. Notice that the two coefficient bundles are isomorphic. According to Section 1.2, ∂_Ω^e admits a K-theory index $\mathrm{Ind}_{V_\Omega} \partial_{\Omega,\mathbb{R}^p}^e$ which belongs to the K-theory group $K_0(C^*(\mathcal{G})) \simeq K_0(C(\Omega) \rtimes \mathbb{Z}^p)$. It is easy to check that the map $e \mapsto \mathrm{Ind}_{V_\Omega}(\partial_\Omega^e)$ induces a well defined morphism from the K-theory of V_Ω to the K-theory of the C^*-algebra $C(\Omega) \rtimes \mathbb{Z}^p$.

Lemma 4.
The longitudinal index of the operator ∂_Ω^e can be represented by the unbounded Kasparov $K-$cycle $(\epsilon^+, \epsilon^-, T_e)$ given by:

- *$\epsilon^\pm$ is the $\mathbb{Z}_2$-graded $C(\Omega) \rtimes \mathbb{Z}^p-$Hilbert module completion of $\tilde{e}[C_c^{\infty,0}(\Omega \times \mathbb{R}^p, S^\pm)^n]$ for the inner product:*
$$\langle \sigma_\pm, \sigma'_\pm \rangle(\omega, k) = \langle \sigma_\pm(\omega, \cdot), k(\sigma'_\pm)(\omega, \cdot) \rangle;$$
- *The right action of $C(\Omega) \rtimes \mathbb{Z}^p$ is given by*
$$\sigma_\pm \cdot h(\omega, t) = \sum_{k \in \mathbb{Z}^p} \sigma_\pm(k\omega, t + k) h(k\omega, k)$$
- *The operator $T_e : \epsilon^+ \to \epsilon^-$ is induced by $\tilde{e} \circ (Id_{C(\Omega)} \otimes \partial)$ on $\tilde{e}[C_c^{\infty,0}(\Omega \times \mathbb{R}^p, S^+)^n]$.*

Proof. The space Ω can be viewed as a transversal of the groupoid $\mathcal{G}$ via the inclusion $\omega \ni \Omega \hookrightarrow (\omega, 0) \in V_\Omega$. The restriction of the groupoid $\mathcal{G}$ to this transversal is thus $\{[\omega, n, n'] \in \Omega \times \mathbb{Z}^p \times \mathbb{Z}^p\}$. This restricted groupoid is then isomorphic to the crossed product groupoid $\Omega \rtimes \mathbb{Z}^p$ via $(\omega, n, n') \to ((-n)(\omega), n' - n)$

The Morita equivalence between $C^*(\mathcal{G})$ and the C^*-algebra $C(\Omega)\rtimes\mathbb{Z}^p$ is implemented by the imprimitivity bimodule ϵ^Ω which is the completion of the algebra $C_c^{\infty,0}(\Omega\times\mathbb{R}^p)$ of compactly supported continuous, smooth in the $\mathbb{R}^p-$variable, functions on $\Omega\times\mathbb{R}^p$, with respect to the $C(\Omega)\rtimes\mathbb{Z}^p$-valued inner product

$$\langle\xi,\xi'\rangle(\omega,k):=\langle\xi(\omega,\cdot),k(\xi')(\omega,\cdot)\rangle,\quad \xi,\xi'\in C_c^{\infty,0}(\Omega\times\mathbb{R}^p)$$

The left action of $C^*(\mathcal{G})$ on ϵ^Ω is given for any $f\in C_c^{\infty,0}(\mathcal{G})$ and $\xi\in C_c^{\infty,0}(\Omega\times\mathbb{R}^p)$ by

$$f\cdot\xi(\omega,t)=\int_{\mathbb{R}^p}f(\omega,t,s)\xi(\omega,s)ds.$$

The right action of $C(\Omega)\rtimes\mathbb{Z}^p$ is given for any $\xi\in C_c^{\infty,0}(\Omega\times\mathbb{R}^p)$ and any $h\in C_c(\Omega\times\mathbb{Z}^p)$ by

$$\xi h(\omega,t)=\sum_{n\in\mathbb{Z}^p}\xi(n\omega,t+n)h(n\omega,n)\,.$$

The tensor product over $C^*(\mathcal{G})$ of $\epsilon_E^\pm$ and ϵ^Ω is then isomorphic to the Hilbert $C(\Omega)\rtimes\mathbb{Z}^p$-module completion of $\tilde{e}(C_c(\Omega\times\mathbb{R}^p,S^\pm)^n)$ for the inner product and right action announced, this isomorphism being induced by

$$\sigma_\pm\otimes\xi\longmapsto\left[(\omega,t)\longmapsto\sigma_\pm\cdot\xi(\omega,t)=\int_{\mathbb{R}^p}\sigma_\pm[\omega,t,s]\xi(\omega,s)ds\right],$$

where, if we denote by ${S_{E,\Omega}}^\pm$ the coefficient bundles of the longitudinal differential operator ∂_Ω^e :

- $\sigma_\pm$ is a section in $C_c^{\infty,0}(\mathcal{G},r^*{S_{E,\Omega}}^\pm)$ smooth in the $\mathbb{R}^p\times\mathbb{R}^p-$variable, and
- ξ is a function of $C_c^{\infty,0}(\Omega\times\mathbb{R}^p)$.

In particular, elements $\sigma_\pm\cdot\xi$ form a dense subspace of $e(C_c(\Omega\times\mathbb{R}^p,S^\pm)^n)$. Moreover, since the operator on $\epsilon_E^\pm\otimes_{C^*(\mathcal{G})}\epsilon^\Omega$ is obtained by tensoring ∂_Ω^e with the identity of ϵ^Ω, the image of $\sigma_\pm\cdot\xi$ under this operator, and through the above isomorphism, is $(\partial_\Omega^e\sigma_+)\cdot\xi=\partial_\Omega^e(\sigma_+\cdot\xi)$. ☐

Theorem 1. *The map $e\mapsto\mathrm{Ind}_{V_\Omega}\partial_{\Omega,\mathbb{R}^p}^e$ induces an isomorphism:*

$$\mu_\Omega^{\mathbb{Z}^p}:K_0(C(V_\Omega))\xrightarrow{\simeq}K_0(C(\Omega)\rtimes\mathbb{Z}^p).$$

Proof. The action of $\mathbb{Z}^p$ on $\Omega\times\mathbb{R}^p$ is free and proper so that the two C^*-algebras $C(V_\Omega)$ and $C_0(\Omega\times\mathbb{R}^p)\rtimes\mathbb{Z}^p$ are Morita equivalent. Hence we have an isomorphism

$$\delta:K_0(C(V_\Omega))\longrightarrow K_0(C_0(\Omega\times\mathbb{R}^p)\rtimes\mathbb{Z}^p).$$

More precisely, the isomorphism δ is the cup product by the class of the Kasparov $K-$cycle $(\epsilon,\phi,0)$ in $KK(C(V_\Omega),C_0(\Omega\times\mathbb{R}^p)\rtimes\mathbb{Z}^p)$, where [51, subsection 4.3]:

- ϵ is the bimodule completion of $C_c(\Omega\times\mathbb{R}^p)$ for the $C_0(\Omega\times\mathbb{R}^p)\rtimes\mathbb{Z}^p$-valued inner product:
$$\langle f,g\rangle(\omega,t,k):=\overline{f}(\omega,t)k(g)(\omega,t)$$
- The $C_0(\Omega\times\mathbb{R}^p)\rtimes\mathbb{Z}^p$-right module structure is given for $f\in C_c(\Omega\times\mathbb{R}^p)$ and $h\in C_c(\Omega\times\mathbb{R}^p\times\mathbb{Z}^p)$ by
$$f\cdot h\,(\omega,t)=\sum_{k\in\mathbb{Z}^p}f(k\omega,kt)h(k\omega,kt,k).$$
- ϕ is the obvious action of $C(V_\Omega)$ as bounded functions on $\mathbb{R}^p\times\Omega$.

On the other hand, denote by 1_Ω the class in $KK^{\mathbb{Z}^p}(C(\Omega),C(\Omega))$ which corresponds to the identity homomorphism $\mathrm{Id}_{C(\Omega)}:C(\Omega)\to C(\Omega)$. The cup product

$$\tau_\Omega[\partial]:=1_\Omega\otimes_{\mathbb{C}}[\partial],$$

of 1_Ω by the class of the Dirac operator on $\mathbb{R}^p$, belongs to $KK^{\mathbb{Z}^p}(C_0(\Omega\times\mathbb{R}^p),C(\Omega))$. If $J_{\mathbb{Z}^p}$ is now the Kasparov transform [39]:

$$J_{\mathbb{Z}^p}:KK^{\mathbb{Z}^p}(C_0(\Omega\times\mathbb{R}^p),C(\Omega))\longrightarrow KK(C_0(\Omega\times\mathbb{R}^p)\rtimes\mathbb{Z}^p,C(\Omega)\rtimes\mathbb{Z}^p)$$

then the corresponding class $J_{\mathbb{Z}^p}(\tau_\Omega[\partial])$ in $KK(C_0(\Omega\times\mathbb{R}^p)\rtimes\mathbb{Z}^p, C(\Omega)\rtimes\mathbb{Z}^p)$ enables to define the Dirac morphism β as the cup product by $J_{\mathbb{Z}^p}(\tau_\Omega([\partial]))$:

$$\beta = \otimes_{C_0(\Omega\times\mathbb{R}^p)\rtimes\mathbb{Z}^p} J_{\mathbb{Z}^p}(\tau_\Omega[\partial]) : K_0(C_0(\Omega\times\mathbb{R}^p)\rtimes\mathbb{Z}^p)) \longrightarrow K_0(C(\Omega)\rtimes\mathbb{Z}^p).$$

The element $J_{\mathbb{Z}^p}(\tau_\Omega[\partial])$ is represented by the Kasparov $K-$cycle:

$$([C(\Omega)\otimes L^2(\mathbb{R}^p, S^+)]\rtimes\mathbb{Z}^p, [C(\Omega)\otimes L^2(\mathbb{R}^p, S^-)]\rtimes\mathbb{Z}^p, \phi, \mathrm{Id}_{C(\Omega)}\otimes\partial),$$

where

$$\mathrm{Id}_{C(\Omega)}\otimes\partial : [C(\Omega)\otimes L^2(\mathbb{R}^p, S^+)]\rtimes\mathbb{Z}^p \longrightarrow [C(\Omega)\otimes L^2(\mathbb{R}^p, S^-)]\rtimes\mathbb{Z}^p,$$

is defined by inflating ∂, and ϕ is the obvious representation of $C_0(\Omega\times\mathbb{R}^p)\rtimes\mathbb{Z}^p$.

Now, because our group $\mathbb{Z}^p$ satisfies the Baum-Connes conjecture (the Kasparov γ-element is trivial for $\mathbb{Z}^p$), the dual-Dirac construction furnishes an inverse for β. In particular we know that β is an isomorphism (see [39]).

The composite map $\beta\circ\delta$ is then an isomorphism that we now identify with the above index map $\mu_\Omega^{\mathbb{Z}^p}$.

Given a projection $e\in M_n(C(V_\Omega))$, the tensor product over $C_0(\Omega\times\mathbb{R}^p)\rtimes\mathbb{Z}^p$ of $e((C_0(\Omega\times\mathbb{R}^p)\rtimes\mathbb{Z}^p)^n)$ by $[C(\Omega)\otimes L^2(\mathbb{R}^p, S^\pm)]\rtimes\mathbb{Z}^p$ coincides with the $C(\Omega)\rtimes\mathbb{Z}^p$-Hilbert module completion of $e(C_c(\Omega\times\mathbb{R}^p, S^\pm)^n)$ for the inner product:

$$\langle\sigma,\sigma'\rangle(\omega,k) = \langle\sigma(\omega,\cdot), k(\sigma'(\omega,\cdot))\rangle,$$

where $k(\sigma')$ is the translated of σ' by k. The operator being then given by

$$\sigma\longmapsto e[(Id_{C(\Omega)}\otimes\partial)\cdot\sigma)].$$

The proof is complete thanks to the description of the index map given in Lemma 4. □

2. Longitudinal de Rham cohomology

The goal of the present section is to define and study the longitudinal de Rham cohomology. This cohomology will be the receptacle for the longitudinal Chern character and allows to state the index theorem.

2.1. The longitudinal de Rham complex.

In this subsection, we describe the longitudinal de Rham cohomology of the foliated mapping torus V_Ω. Since the relative case will also be needed, we also give the corresponding definitions for relative open pairs of V_Ω. Let $\Omega_b^k(\mathbb{R}^p)$ be the space of k-differential forms on the vector space $\mathbb{R}^p$ with bounded derivatives of all orders.

Definition 3. A (real) longitudinal differential k-form on V_Ω is a $\mathbb{Z}^p$-equivariant continuous map $\phi:\Omega\to\Omega_b^k(\mathbb{R}^p)$.

We denote by $\Omega_\ell^k(V_\Omega,\mathbb{R})$ the space of longitudinal differential $k-$forms on V_Ω. If ϕ is a longitudinal $k-$form, its longitudinal differential $d_\ell(\phi)$ is by definition the longitudinal $(k+1)$-differential form which is given by the $\mathbb{Z}^p$-equivariant map $\omega\mapsto d(\phi(\omega,\cdot))$ where d is the de Rham differential on $\mathbb{R}^p$. It belongs to $\Omega_\ell^{k+1}(V_\Omega,\mathbb{R})$ as can be checked easily. So:

$$d_\ell : \Omega_\ell^k(V_\Omega,\mathbb{R}) \longrightarrow \Omega_\ell^{k+1}(V_\Omega,\mathbb{R})$$

provides a differential structure on the graded vector space $\Omega_\ell^*(V_\Omega,\mathbb{R}) = \bigoplus\Omega_\ell^k(V_\Omega,\mathbb{R})$ and satisfies $d_\ell\circ d_\ell = 0$. The cohomology of the complex $(\Omega_\ell^*(V_\Omega,\mathbb{R}), d_\ell)$ will be denoted by

$$\mathrm{H}_\ell^*(V_\Omega,\mathbb{R}) = \bigoplus_{k\geq 0}\mathrm{H}_\ell^k(V_\Omega,\mathbb{R}).$$

Remark 1. It is proved in [47] that $\mathrm{H}_\ell^*(V_\Omega,\mathbb{R})$ is also the cohomology of the sheaf of continuous functions which are locally constant in the leaf direction. This sheaf is, in our situation, the sheaf of continuous and equivariant functions on an equivariant open subset of $\Omega\times\mathbb{R}^p$, constant in the $\mathbb{R}^p-$direction. If $\mathrm{H}^*(V_\Omega,\mathbb{R})$ denotes the Čech cohomology groups of V_Ω with real coefficients, we then have a well defined morphism:

$$\mathrm{H}^*(V_\Omega,\mathbb{R}) \longrightarrow \mathrm{H}_\ell^*(V_\Omega,\mathbb{R})$$

induced by the natural morphism of sheaves.

If ϕ is a longitudinal differential form, the support of ϕ is by definition the image of the support of $(\omega, t) \mapsto \phi(\omega, t)$ under the projection $\Omega \times \mathbb{R}^p \to V_\Omega$. The support of ϕ is thus a compact subset of V_Ω. Let now U be an open subset of V_Ω and let $\Omega^*_{\ell,c}(U,\mathbb{R}) = \bigoplus \Omega^k_{\ell,c}(U,\mathbb{R})$ be the space of longitudinal differential $k-$forms with support in U. The restriction of the longitudinal differential d_ℓ to $\Omega^*_{\ell,c}(U,\mathbb{R})$ preserves it and we get a subcomplex $(\Omega^*_{\ell,c}(U,\mathbb{R}), d_\ell)$ of the longitudinal complex $(\Omega^*_\ell(V_\Omega,\mathbb{R}), d_\ell)$. We denote by

$$\mathrm{H}^*_{\ell,c}(U,\mathbb{R}) = \bigoplus \mathrm{H}^k_{\ell,c}(U,\mathbb{R})$$

the cohomology of this subcomplex.

We shall also use the relative longitudinal cohomology for open pairs, i.e. pairs (U_0, U_1) of open subsets of V_Ω such that $U_0 \subset U_1$.

Definition 4. The relative longitudinal complex associated with an open pair (U_0, U_1), denoted $\Omega^*_{\ell,c}(U_0, U_1, \mathbb{R})$, is the quotient complex

$$0 \to \Omega^*_{\ell,c}(U_0,\mathbb{R}) \to \Omega^*_{\ell,c}(U_1,\mathbb{R}) \to \Omega^*_{\ell,c}(U_0,U_1,\mathbb{R}) \to 0.$$

The cohomology of this complex is called the relative longitudinal cohomology for the pair (U_0, U_1) and is denoted by $\mathrm{H}^*_{\ell,c}(U_0,U_1,\mathbb{R})$.

From classical homological algebra, we get:

Proposition 2. *The following long exact sequence holds:*

$$\cdots \to \mathrm{H}^k_{\ell,c}(U_0,\mathbb{R}) \to \mathrm{H}^k_{\ell,c}(U_1,\mathbb{R}) \to \mathrm{H}^k_{\ell,c}(U_0,U_1,\mathbb{R}) \to \mathrm{H}^{k+1}_{\ell,c}(U_0,\mathbb{R}) \to \cdots$$

2.2. **The top dimensional longitudinal cohomology.** We end this section by studying the top dimension group of longitudinal cohomology of V_Ω. In particular, we indentify this group with the coinvariants of the real valued continuous functions on Ω for the action of $\mathbb{Z}^p$ (Theorem 2).

Proposition 3. *For any $\phi \in \Omega^p_\ell(V_\Omega,\mathbb{R})$, we define a continuous function $\Psi_{\mathbb{Z}^p}(\phi)$ on Ω by setting:*

$$\Psi_{\mathbb{Z}^p}(\phi)(\omega) := \int_{]0,1[^p} \phi(\omega, x), \quad \omega \in \Omega.$$

Then $\Psi_{\mathbb{Z}^p}$ induces a map from $\mathrm{H}^p_\ell(V_\Omega,\mathbb{R})$ to the coinvariants $C(\Omega,\mathbb{R})_{\mathbb{Z}^p}$ of the algebra $C(\Omega,\mathbb{R})$ of continuous real valued function on Ω by the action of $\mathbb{Z}^p$ i.e. the quotient of $C(\Omega,\mathbb{R})$ by the subspace generated by elements $g - n(g)$, where $g \in C(\Omega,\mathbb{R})$ and $n \in \mathbb{Z}^p$.

Proof. We need to show that the image of an exact longitudinal differential form of degree p is in the subspace generated by elements of the form $n(g) - g$ where $n \in \mathbb{Z}^p$ and $g \in C(\Omega)$. Actually, it is enough to check it for longitudinal differential forms $d_\ell(\phi)$ where

$$\phi(\omega, t_1, \cdots, t_p) = f(\omega, t_1, \cdots, t_p) dt_1 \cdots \hat{dt_j} \cdots dt_p,$$

and f is a $\mathbb{Z}^p$-invariant function on $\Omega \times \mathbb{R}^p$ which is smooth in the $\mathbb{R}^p$-direction. But then,

$$\Psi_{\mathbb{Z}^p}(d_\ell \phi)(\omega) = \int_{]0,1[^p} (\partial_{t_j} f)(\omega, t_1, \cdots, t_p) dt_1 \cdots dt_p,$$

and with respect to t_j, the integral gives:

$$f(\omega, t_1, \cdots, t_{j-1}, 1, t_{j+1}, \cdots, t_p) - f(\omega, t_1, \cdots, t_{j-1}, 0, t_{j+1}, \cdots, t_p).$$

Hence, if we denote by g the function on Ω defined by:

$$g(\omega) := \int_{]0,1[^{p-1}} f(\omega, t_1, \cdots, t_{j-1}, 0, t_{j+1}, \cdots, t_p) dt_1 \cdots \hat{dt_j} \cdots dt_p,$$

then we obtain:

$$\Psi_{\mathbb{Z}^p}(d_\ell \phi) = e^j(g) - g,$$

with e^j being the j^{th} vector of the canonical basis of $\mathbb{Z}^p$. $\square$

Theorem 2. *The transform $\psi_{\mathbb{Z}^p}$ is an isomorphism, i.e.* $\mathrm{H}_\ell^p(V_\Omega,\mathbb{R}) \cong C(\Omega,\mathbb{R})_{\mathbb{Z}^p}$.

Proof. Let φ be a smooth compactly supported function on $]0,1[$ such that:

$$\int_{]0,1[} \varphi(s)ds = 1.$$

The space $\Omega_\ell^p(V_\Omega,\mathbb{R})$ is composed of differential forms $f dt_1\wedge\cdots\wedge dt_p$ where f is a $\mathbb{Z}^p-$invariant, continuous smooth in the $\mathbb{R}^p-$direction function on $\Omega\times\mathbb{R}^p$. We define the operator K from $\Omega_\ell^p(V_\Omega,\mathbb{R})$ to $\Omega_\ell^{p-1}(V_\Omega,\mathbb{R})$ by the following formula:

$$K(fdt_1\wedge\cdots\wedge dt_p)(\omega,t_1,\cdots,t_p) := \tag{2}$$
$$\sum_{j=1}^{p}(-1)^{j-1}\Bigg[\int_{]0,t_j[\times]0,1[^{p-j}} f(\omega,t_1,\cdots,t_{j-1},s_j,\cdots,s_p)ds_j\cdots ds_p -$$
$$\left(\int_{]0,1[^{p-j+1}} f(\omega,t_1,\cdots,t_{j-1},s_j,\cdots,s_p)ds_j\cdots ds_p\right)\times\left(\int_{]0,t_j[}\varphi(s)ds\right)\Bigg]$$
$$\varphi(t_{j+1})\cdots\varphi(t_p)dt_1\wedge\cdots\wedge\hat{dt_j}\wedge\cdots dt_p,$$

if $(t_1,\cdots,t_p)\in]0,1]^p$ and we extend this form to $\mathbb{R}^p$ by requiring $\mathbb{Z}^p-$invariance. We claim that $K(fdt_1\wedge\cdots\wedge dt_p)$ belongs to $\Omega_\ell^{p-1}(V_\Omega,\mathbb{R})$. Let us fix an integer $k\in\{1,\cdots,p\}$. As before, we denote by e^k be the k^{th} vector of the canonical basis of $\mathbb{Z}^p$. Then we must show that if $(t_1,\cdots,t_{k-1},t_{k+1}\cdots,t_p)\in]0,1[^{p-1}$, we have:

$$K(fdt_1\wedge\cdots\wedge dt_p)(e^k(\omega),t_1,\cdots,t_{k-1},1,t_{k+1}\cdots,t_p) = K(fdt_1\wedge\cdots\wedge dt_p)(\omega,t_1,\cdots,t_{k-1},0,t_{k+1}\cdots,t_p).$$

This is achieved by direct inspection. If $j>k$, then the invariance of f gives immediately the equality of the terms corresponding to j in the expression of the sum (2). If $j<k$, then since φ is compactly supported in $]0,1[$, the terms corresponding to j in the expression of the sum (2) for $(e^k(\omega),t_1,\cdots,t_{k-1},1,t_{k+1}\cdots,t_p)$ and for $(\omega,t_1,\cdots,t_{k-1},0,t_{k+1}\cdots,t_p)$ are equal to zero.

Now if $j=k$, then

$$\int_{]0,t_j=1[\times]0,1[^{p-j}} f(\omega,t_1,\cdots,t_{j-1},s_j,\cdots,s_p)ds_j\cdots ds_p -$$
$$\left(\int_{]0,1[^{p-j+1}} f(\omega,t_1,\cdots,t_{j-1},s_j,\cdots,s_p)ds_j\cdots ds_p\right)\times\left(\int_{]0,t_j=1[}\varphi(s)ds\right)=0$$

because $\int_{]0,1[}\varphi(s)ds=1$. Therefore the operator K sends $\Omega_\ell^p(V_\Omega,\mathbb{R})$ to $\Omega_\ell^{p-1}(V_\Omega,\mathbb{R})$ as claimed.

Let $\Lambda: C(\Omega)_{\mathbb{Z}^p}\to \mathrm{H}_\ell^p(V_\Omega,\mathbb{R})$ be mapping the class of $g\in C(\Omega,\mathbb{R})$ in $C(\Omega,\mathbb{R})_{\mathbb{Z}^p}$ to the cohomological class of the longitudinal p-form defined by: $(t_1,\cdots,t_p)\mapsto g(\omega)\varphi(t_1)\cdots\varphi(t_p)dt_1\wedge\cdots\wedge dt_p$, if $(t_1,\cdots,t_p)\in]0,1[^p$ and extended to $\mathbb{R}^p$ by requiring $\mathbb{Z}^p-$invariance. Assume that $g=e^j(f)-f$ where $f\in C(\Omega,\mathbb{R})$ and $e^j\in\mathbb{Z}^p$ is as above the j^{th} vector of the canonical basis of $\mathbb{Z}^p$. Then $\Lambda(g)=d_\ell(\alpha)$ with

$$\alpha(\omega,t_1,\cdots,t_p)=\varphi(t_1)\cdots\varphi(t_{j-1})\left(f((-e_i)(\omega))\int_{]0,t_j[}\varphi(s)d_s+f(\omega)\int_{]t_j,1[}\varphi(s)d_s\right)$$
$$\varphi(t_{j+1})\cdots\varphi(t_p)dt_1\wedge\cdots\wedge dt_{j-1}\wedge dt_{j+1}\wedge\cdots\wedge dt_p$$

Therefore Λ is well defined.

Now, a straightforward computation of $d_\ell\circ K$ yields the relation:

$$d_\ell\circ K = id-\Lambda\circ\psi_{\mathbb{Z}^p}.$$

On the other hand, $\psi_{\mathbb{Z}^p}\circ\Lambda$ is equal to the identity map. So the proof of the theorem is complete and we get, as a byproduct, that the cohomology map induced by Λ does not depend on the choice of the function φ. ∎

Remark 2. The abstract isomorphism in the above theorem is a consequence of the spectral sequence computation carried out in the last section of this paper. However, we shall need a concrete realization of this Poincaré map.

3. Cyclic homology for quasi-crystals

This section is devoted to the study of the periodic cyclic homology for the algebras we are interested in. We take this opportunity to identify the periodic cyclic (co)homology of the mapping torus of the action of $\mathbb{Z}^p$ on the Cantor set Ω with the group (co)homology of $\mathbb{Z}^p$ with coefficients in the $\mathbb{Z}^p$-module $C(\Omega)$ (or the space of measures $\mathcal{M}(\Omega)$, for cohomology). The Chern-Connes character is also defined and studied.

3.1. **Review of cyclic homology.** For the benefit of the reader, we shall briefly review some definitions and gather some well known properties of cyclic homology. For the sake of simplicity, we shall restrict ourselves to the algebraic context and we refer for instance to [12, 13, 17, 25, 26, 49, 37, 57, 59] for the corresponding statements in the appropriate topological framework.

Let $\mathcal{A}$ be a given algebra over the field $\mathbb{C}$ of complex numbers. Denote by $\tilde{\mathcal{A}}$ the unitalization of $\mathcal{A}$. Consider the mixed complex $(\mathcal{C}_*(\mathcal{A}), \tilde{b}, \tilde{B})$ where:

$$\mathcal{C}_k(\mathcal{A}) := \tilde{\mathcal{A}} \otimes \mathcal{A}^{\otimes k} \simeq \mathcal{A}^{\otimes k+1} \oplus \mathcal{A}^{\otimes k}, \;\; \tilde{b} = \begin{pmatrix} b & 1-\lambda \\ 0 & -b' \end{pmatrix} \text{ and } \tilde{B} = \begin{pmatrix} 0 & 0 \\ N & 0 \end{pmatrix},$$

and the operators b' and b are defined by:

$$b'(a^0 \otimes a^1 \otimes \cdots \otimes a^k) := \sum_{j=0}^{k-1} (-1)^j a^0 \otimes \cdots \otimes a^j a^{j+1} \otimes a^{j+2} \otimes \cdots \otimes a^k \text{ and}$$
$$b(a^0 \otimes a^1 \otimes \cdots \otimes a^k) = b'(a^0 \otimes a^1 \otimes \cdots \otimes a^k) + (-1)^k a^k a^0 \otimes a^1 \otimes \cdots \otimes a^{k-1}.$$

The operators λ and N are defined by:

$$\lambda(a_0 \otimes \cdots \otimes a_k) := (-1)^k a_k \otimes a_0 \otimes \cdots \otimes a_{k-1} \text{ and } N := id + \lambda + \cdots + \lambda^{k-1}.$$

The relations $b^2 = 0$, $b'^2 = 0$ and $b(1-\lambda) = (1-\lambda)b'$ then show that:

$$\tilde{b}^2 = \tilde{B}^2 = \tilde{b}\tilde{B} + \tilde{B}\tilde{b} = 0.$$

The periodic cyclic homology is defined using the $\mathbb{Z}^2$-graded complex:

$$\cdots \to \mathcal{C}P^0(\mathcal{A}) \xrightarrow{\tilde{b}+\tilde{B}} \mathcal{C}P^1(\mathcal{A}) \xrightarrow{\tilde{b}+\tilde{B}} \mathcal{C}P^0(\mathcal{A}) \to \cdots,$$

where $\mathcal{C}P^0(\mathcal{A}) := \Pi_k \mathcal{C}_{2k}(\mathcal{A})$ and $\mathcal{C}P^1(\mathcal{A}) := \Pi_k \mathcal{C}_{2k+1}(\mathcal{A})$. The definitions simplify when the algebra $\mathcal{A}$ is H-unital (e.g. when $\mathcal{A}$ is unital), see for instance [18]. In particular, one is then led to the operators b and B.

Let us fix now the definition of the Chern-Connes character that will be used later on. Our references are [19, 31, 44]. If $e = (e_{ij})_{1\le i,j\le n}$ is an idempotent in the matrix algebra $M_n(\tilde{\mathcal{A}})$, then we set for any $k \ge 1$:

$$\mathrm{Ch}_k(e) := \frac{(-1)^k (2k)!}{k!} \sum_{1 \le i_0, \cdots, i_{2k} \le n} (e_{i_0 i_1} - \frac{\delta_{i_0 i_1}}{2}) \otimes e_{i_1 i_2} \otimes \cdots \otimes e_{i_{2k-1} i_{2k}} \otimes e_{i_{2k} i_0} \quad \in \mathcal{C}_{2k}(\mathcal{A}),$$

where $\delta_{i_0 i_1}$ is the Kronecker symbol, equal to 0 if $i_0 \neq i_1$ and to 1 if $i_0 = i_1$. For $k = 0$, we set $\mathrm{Ch}_0(e) := \sum_{i=1}^n e_{ii}$. In the odd case, we can define similarly the Chern-Connes character by setting, when $\mathcal{A}$ is unital for simplicity, and for any invertible matrix $u = (u_{ij})_{1\le i,j\le n}$ in $M_n(\mathcal{A})$:

$$\mathrm{Ch}_k(u) := (-1)^k k! \sum_{1 \le i_0, \cdots, i_{2k+1} \le n} u_{i_0 i_1} \otimes u^{-1}_{i_1 i_2} \otimes \cdots \otimes u^{-1}_{i_{2k+1} i_0} \in \mathcal{C}_{2k+1}(\mathcal{A}), \quad k \ge 0.$$

In the present paper, we shall be dealing with algebras with finite Hochschild dimension and it will always be possible to see $\mathrm{Ch}_*(e)$ and $\mathrm{Ch}_*(u)$ as periodic cyclic homology classes in $\mathrm{HP}_*(\mathcal{A})$. If e and e' are two idempotents in $M_n(\tilde{\mathcal{A}})$ such that $x = [e] - [e'] \in K_0(\mathcal{A})$, then the class $[\mathrm{Ch}_*(e)] - [\mathrm{Ch}_*(e')]$ in $\mathrm{HP}_0(\mathcal{A})$ only depends on the K-theory class x, see for instance [31][Proposition 1.1 and Proposition 1.3]. Hence we obtain a group homomorphism:

$$\mathrm{Ch} : K_0(\mathcal{A}) \longrightarrow \mathrm{HP}_0(\mathcal{A}),$$

called the Chern-Connes character. In the same way, we get a well defined Chern-Connes character from odd $K-$theory to odd periodic cyclic homology, see also [19]. To sum up, we have a $\mathbb{Z}_2-$graded morphism:

$$\mathrm{Ch} : K_i(\mathcal{A}) \longrightarrow \mathrm{HP}_i(\mathcal{A}), \quad i = 0, 1.$$

Let $0 \to I \hookrightarrow \mathcal{A} \longrightarrow \mathcal{A}/I \to 0$ be an extension of algebras. We denote by $\mathcal{C}_*(\mathcal{A}, I)$ the kernel of the surjection $\mathcal{C}_*(\mathcal{A}) \longrightarrow \mathcal{C}_*(\mathcal{A}/I)$. Then $\mathcal{C}_*(\mathcal{A}, I)$ is preserved by the operators $\tilde{b}$ and $\tilde{B}$ on $\mathcal{C}_*(\mathcal{A})$ and we denote by $\mathrm{HP}_*(\mathcal{A}, I)$ the periodic homology of this mixed bicomplex. We get from classical homological algebra [56], a six-term exact sequence

$$\begin{tikzcd}
\mathrm{HP}_0(\mathcal{A}, I) \arrow[r] & \mathrm{HP}_0(\mathcal{A}) \arrow[r] & \mathrm{HP}_0(\mathcal{A}/I) \arrow[d] \\
\mathrm{HP}_1(\mathcal{A}/I) \arrow[u] & \mathrm{HP}_1(\mathcal{A}) \arrow[l] & \mathrm{HP}_1(\mathcal{A}, I) \arrow[l]
\end{tikzcd}$$

On the other hand, the inclusion $\mathcal{C}_*(I) \hookrightarrow \mathcal{C}_*(\mathcal{A}, I)$ induces a well defined $\mathbb{Z}_2-$graded map:

$$i_* : \mathrm{HP}_*(I) \longrightarrow \mathrm{HP}_*(\mathcal{A}, I).$$

Theorem 3. [26, 49] *The map i_* is an isomorphism. Therefore, we have the following six-term exact sequence*

$$\begin{tikzcd}
\mathrm{HP}_0(I) \arrow[r] & \mathrm{HP}_0(\mathcal{A}) \arrow[r] & \mathrm{HP}_0(\mathcal{A}/I) \arrow[d] \\
\mathrm{HP}_1(\mathcal{A}/I) \arrow[u] & \mathrm{HP}_1(\mathcal{A}) \arrow[l] & \mathrm{HP}_1(I) \arrow[l]
\end{tikzcd}$$

Remark 3. Under appropriate topological assumptions, excision also holds for topological homology, see for instance [17, 25, 26].

In order to obtain strict compatibility of the Chern-Connes character with the above exact sequence, one needs to multiply the boundary map $\mathrm{HP}_0(\mathcal{A}/I) \to \mathrm{HP}_1(I)$ by a factor $2i\pi$.

Theorem 4. [25, 49] *The Chern-Connes character is a natural transformation between K-theory and periodic cyclic homology, which commutes with the boundaries of the six term exact sequence in K-theory and those of the above six term exact sequence in periodic cyclic homology.*

The periodic cyclic homology $\mathrm{HP}_*(C^{\infty,0}(V_\Omega))$ of the algebra $C^{\infty,0}(V_\Omega)$ of continuous longitudinally smooth functions on the foliated space V_Ω is easy to compute. It is worthpointing out (see the proof below) that this homology is isomorphic to the periodic cyclic homology of the crossed product algebra $C(\Omega) \rtimes \mathbb{Z}^p$.

Theorem 5. *The periodic cyclic homology of the commutative Frechet algebra $C^{\infty,0}(V_\Omega)$ is isomorphic to the homology of the group $\mathbb{Z}^p$ with coefficients in the $\mathbb{Z}^p$-module $C(\Omega)$. More precisely, we have:*

$$\mathrm{HP}_0(C^{\infty,0}(V_\Omega)) \simeq \bigoplus_{j\geq 0} \mathrm{H}_{2j}(\mathbb{Z}^p, C(\Omega)) \text{ and } \mathrm{HP}_1(C^{\infty,0}(V_\Omega)) \simeq \bigoplus_{j\geq 0} \mathrm{H}_{2j+1}(\mathbb{Z}^p, C(\Omega)).$$

Proof. The proof of this theorem is similar to the computation of the longitudinal cohomology carried out in Section 4. This theorem is essentially a consequence of more general results from [43, 49]. We proceed now to explain the details.

The periodic cyclic homology of $C^{\infty,0}(V_\Omega)$ can be computed using the results of [49]. The first result that we shall use from Nistor's paper is that the periodic cyclic homology is concentrated at the torsion elements so that for the algebra $C_c^{\infty,0}(\mathbb{R}^p \times \Omega) \rtimes \mathbb{Z}^p$, it is isomorphic to the homogeneous part $\mathrm{HP}_*(C_c^{\infty,0}(\mathbb{R}^p \times \Omega) \rtimes \mathbb{Z}^p)_{\{0\}}$

[49]. On the other hand, by using [43][Theorem 4], we deduce that the homogeneous part of the periodic cyclic homology of $C_c^{\infty,0}(\mathbb{R}^p\times\Omega)\rtimes\mathbb{Z}^p$ is isomorphic to the homogeneous part of the periodic cyclic homology of the algebra $C(\Omega)\rtimes\mathbb{Z}^p$. By [49][Theorem 2.6], there exists a spectral sequence $\mathrm{EC}^r_{k,h}$ converging to the homogeneous part $\mathrm{HC}_*(C^{\infty,0}(V_\Omega))_{\{0\}}$ of the cyclic homology, with EC^2-term given by:

$$\mathrm{EC}^2_{k,h}=\mathrm{H}_k(\mathbb{Z}^p,\mathrm{HC}_h(C(\Omega))).$$

Since Ω is totally disconnected and compact, the space $\mathrm{HC}_h(C(\Omega))$ identifies with $C(\Omega)$ for h even and vanishes for h odd. This enables to compute the EC^2-term:

$$\mathrm{EC}^2_{k,h}=\mathrm{H}_k(\mathbb{Z}^p,C(\Omega)) \text{ if } h \text{ is even, and } \mathrm{EC}^2_{k,h}=\{0\} \text{ otherwise.}$$

Now again by [43], the spectral sequence EC^r actually collapses at E^2. We therefore get:

$$\mathrm{HC}_q(C^{\infty,0}(V_\Omega))_{\{0\}}\simeq\oplus_{k+2h=q}\mathrm{H}_k(\mathbb{Z}^p,C(\Omega)).$$

A direct inspection of the operator S then completes the proof, see again [43]. □

3.2. **The longitudinal Chern character.** Recall that by definition, the algebra of longitudinal smooth function $C^{\infty,0}(V_\Omega)$ is $\Omega^0_\ell(V_\Omega,\mathbb{R})\otimes\mathbb{C}$, and that this is a dense subalgebra of $C(V_\Omega)$ stable under holomorphic calculus. In particular, we have an isomorphism $K_*(C^{\infty,0}(V_\Omega))\cong K_*(C(V_\Omega))$ induced by the inclusion. In the same way, if U is an open subset of V_Ω, we denote $\Omega^0_{\ell,c}(U,\mathbb{R})\otimes\mathbb{C}$ by $C_c^{\infty,0}(U)$ and for any open pair (U_0,U_1), we denote $\Omega^0_{\ell,c}(U_0,U_1,\mathbb{R})\otimes\mathbb{C}$ by $C_c^{\infty,0}(U_0,U_1)$.

We now define the longitudinal Chern character:

$$ch_\ell: K_i(C^{\infty,0}(V_\Omega)))\longrightarrow \mathrm{H}^{[i]}_\ell(V_\Omega,\mathbb{R}),\quad i=0,1,$$

where $\mathrm{H}^{[i]}_\ell$ stands for $\oplus_{j\in\mathbb{Z}}\mathrm{H}^{i+2j}_\ell$. In fact, we will need to define more generally a Chern character for every open pair.

Definition 5. The longitudinal Hochschild-Kostant-Rosenberg map $\chi^\ell_n:\mathcal{C}_n(C^{\infty,0}(V_\Omega))\to\Omega^*_\ell(V_\Omega,\mathbb{R})\otimes\mathbb{C}$ is defined by

$$\chi^\ell_n(f^0\hat{\otimes}\cdots\hat{\otimes}f^n):=\left(\frac{1}{2i\pi}\right)^{\left[\frac{n+1}{2}\right]}\frac{1}{n!}f^0d_\ell f^1\wedge\cdots\wedge d_\ell f^n,$$

where $\left[\frac{n+1}{2}\right]$ is the integer part of $(n+1)/2$. The normalization constants are fixed by the Bott element on $\mathbb{R}^n$, see [18] for the even case and [16] for the odd case.

Proposition 4. *We have*

$$\chi^\ell_*\circ b=0,\ \chi^\ell_{2k+1}\circ B=\frac{1}{2i\pi}d_\ell\circ\chi^\ell_{2k}\ \textit{and}\ \chi^\ell_{2k+2}\circ B=d_\ell\circ\chi^\ell_{2k+1}.$$

Proof. We have:

$$(\chi^\ell_{n-1}\circ b)(f^0\hat{\otimes}\cdots\hat{\otimes}f^n)=\left(\frac{1}{2i\pi}\right)^{\left[\frac{n}{2}\right]}\frac{(-1)^{n-1}}{n!}[f^0d_\ell f^1\cdots d_\ell f^{n-1},f^n].$$

But the commutator $[f^0d_\ell f^1\cdots d_\ell f^{n-1},f^n]$ is trivial, since the algebra $\Omega^*_\ell(V_\Omega,\mathbb{R})\otimes\mathbb{C}$ is graded commutative. Hence we deduce the first relation. On the other hand,

$$\begin{aligned}(\chi^\ell_{n+1}\circ B)(f^0\hat{\otimes}\cdots\hat{\otimes}f^n) &= (1/2i\pi)^{\left[\frac{n+2}{2}\right]}\frac{1}{(n+1)!}\sum_{0\le j\le n}(-1)^{nj}d_\ell(f^j)\cdots d_\ell(f^n)d_\ell(f^0)\cdots d_\ell(f^{j-1})\\ &= (1/2i\pi)^{\left[\frac{n+2}{2}\right]}\frac{1}{(n+1)!}\sum_{0\le j\le n}(-1)^{nj}(-1)^{(n-j+1)j}d_\ell(f^0)\cdots d_\ell(f^n)\\ &= (1/2i\pi)^{\left[\frac{n+2}{2}\right]}\frac{1}{n!}d_\ell(f^0)\cdots d_\ell(f^n).\end{aligned}$$

The easy computation of $d_\ell\circ\chi^\ell_n(f^0\hat{\otimes}\cdots\hat{\otimes}f^n)$ then finishes the proof. □

Corollary 1. *The* HKR *map* χ_*^ℓ *induces morphism (that we still denote by* χ_*^ℓ*)*

$$\chi_*^\ell : \mathrm{HP}_i(C^{\infty,0}(V_\Omega))) \longrightarrow \mathrm{H}_\ell^{[i]}(V_\Omega, \mathbb{R}) \otimes \mathbb{C},$$

for $i \in \{0,1\}$*. In the same way, for any open subset* U *of* V_Ω*, the* HKR *map induces by restriction a chain morphism between the periodic complexes for* $C_c^{\infty,0}(U)$ *and the longitudinal de Rham complex*

$$\Omega^e_{\ell,c}(U,\mathbb{R}) \otimes \mathbb{C} \underset{d_\ell}{\overset{\frac{1}{2i\pi}d_\ell}{\leftrightarrows}} \Omega^o_{\ell,c}(U,\mathbb{R}) \otimes \mathbb{C}$$

and thus a morphism

$$\chi_*^\ell : \mathrm{HP}_i(C_c^{\infty,0}(U))) \longrightarrow \mathrm{H}_{\ell,c}^{[i]}(U, \mathbb{R}) \otimes \mathbb{C},$$

for $i \in \{0,1\}$.

We shall also need a relative version of the HKR maps in periodic homology. Let (U_0, U_1) be an open pair of V_Ω. Recall that we have the short exact sequence

$$0 \to C_c^{\infty,0}(U_0) \longrightarrow C_c^{\infty,0}(U_1) \longrightarrow C_c^{\infty,0}(U_0,U_1) \to 0.$$

Then χ_*^ℓ maps the subcomplex $\mathcal{C}_*(C_c^{\infty,0}(U_0))$ of $\mathcal{C}_*(C_c^{\infty,0}(U_1))$ to the subcomplex $\Omega^*_{\ell,c}(U_0,\mathbb{R})\otimes\mathbb{C}$ of $\Omega^*_{\ell,c}(U_1,\mathbb{R})\otimes\mathbb{C}$ and thus induces a chain morphism (up to rescaling the odd differential) from $\mathcal{C}_*(C_c^{\infty,0}(U_1))/\mathcal{C}_*(C_c^{\infty,0}(U_0))$ to $\Omega^*_{\ell,c}(U_0,U_1,\mathbb{R}) \otimes \mathbb{C} = (\Omega^*_{\ell,c}(U_1,\mathbb{R})/\Omega^*_{\ell,c}(U_0,\mathbb{R})) \otimes \mathbb{C}$. Algebras like $C_c^{\infty,0}(U)$ for an open subset U of V_Ω satisfy topological excision [26, 49]. Therefore the chain morphism

$$\mathcal{C}_*(C_c^{\infty,0}(U_1))/\mathcal{C}_*(C_c^{\infty,0}(U_0)) \longrightarrow \mathcal{C}_*(C_c^{\infty,0}(U_0,U_1))$$

is a quasi isomorphism and thus we get

Proposition 5. *Let* (U_0, U_1) *be an open pair of* V_Ω*. Then there is a relative HKR morphism in periodic homology*

$$\chi_*^\ell : \mathrm{HP}_*(C_c^{\infty,0}(U_0,U_1)) \longrightarrow \mathrm{H}_{\ell,c}^{[*]}(U_0,U_1,\mathbb{R}) \otimes \mathbb{C},$$

such that the both following six-term exact sequences

$$\begin{tikzcd}
\mathrm{HP}_0(C_c^{\infty,0}(U_0)) \arrow[r] & \mathrm{HP}_0(C_c^{\infty,0}(U_1)) \arrow[r] & \mathrm{HP}_0(C_c^{\infty,0}(U_0,U_1)) \arrow[d] \\
\mathrm{HP}_1(C_c^{\infty,0}(U_0,U_1)) \arrow[u] & \mathrm{HP}_1(C_c^{\infty,0}(U_1)) \arrow[l] & \mathrm{HP}_1(C_c^{\infty,0}(U_0)) \arrow[l]
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathrm{H}_{\ell,c}^{[0]}(U_0,\mathbb{R}) \otimes \mathbb{C} \arrow[r] & \mathrm{H}_{\ell,c}^{[0]}(U_1,\mathbb{R}) \otimes \mathbb{C} \arrow[r] & \mathrm{H}_{\ell,c}^{[0]}(U_0,U_1,\mathbb{R}) \otimes \mathbb{C} \arrow[d] \\
\mathrm{H}_{\ell,c}^{[1]}(U_0,U_1,\mathbb{R}) \otimes \mathbb{C} \arrow[u] & \mathrm{H}_{\ell,c}^{[1]}(U_1,\mathbb{R}) \otimes \mathbb{C} \arrow[l] & \mathrm{H}_{\ell,c}^{[1]}(U_0,\mathbb{R}) \otimes \mathbb{C} \arrow[l]
\end{tikzcd}$$

are intertwined by the HKR maps, where again the boundary map from HP_0 *to* HP_1 *is multiplied by* $2i\pi$.

Definition 6. The longitudinal Chern character $ch_\ell : K_0(C(V_\Omega)) \cong K_0(C^{\infty,0}(V_\Omega)) \to \mathrm{H}_\ell^{[0]}(V_\Omega,\mathbb{R})$, is by definition the composite map

$$K_0(C^{\infty,0}(V_\Omega)) \xrightarrow{\mathrm{Ch}} \mathrm{HP}_0(C^{\infty,0}(V_\Omega)) \xrightarrow{\chi_*^\ell} \mathrm{H}_\ell^{[0]}(V_\Omega,\mathbb{R}) \otimes \mathbb{C}$$

(we will see below that the range of this composition lies in fact in $\mathrm{H}_\ell^{[0]}(V_\Omega,\mathbb{R})$).

An easy consequence of this definition is that for any projection e in $M_n(C^{\infty,0}(V_\Omega))$, $ch_\ell([e])$ is the longitudinal cohomology class of the closed longitudinal differential form $\mathrm{Tr}\left[(e-1/2)\exp\left(\dfrac{-d_\ell e d_\ell e}{2i\pi}\right)\right]$. Indeed, using the involution $u=2e-1$ which anticommutes with $d_\ell e$, one easily deduces that the longitudinal differential form $\mathrm{Tr}\left[\exp\left(\frac{d_\ell e d_\ell e}{2i\pi}\right)\right]$ is a coboundary, while $\mathrm{Tr}\left[d_\ell e\exp\left(\frac{d_\ell e d_\ell e}{2i\pi}\right)\right]$ is trivial. Thus, we can define $ch_\ell([e])$ as the longitudinal cohomology class of $\mathrm{Tr}\left[e\exp\left(\dfrac{-d_\ell e d_\ell e}{2i\pi}\right)\right]$. In particular, since e can be choosen self-adjoint, the range of the longitudinal Chern character lies in $\mathrm{H}_\ell^{[0]}(V_\Omega,\mathbb{R})$. In the same way, the longitudinal Chern character $ch_\ell : K_0(C_0(U))\cong K_0(C_c^{\infty,0}(U))\to \mathrm{H}_{\ell,c}^{[0]}(U,\mathbb{R})$, can be defined for every open subset U of V_Ω by using the following composite map

$$K_0(C_c^{\infty,0}(U))\xrightarrow{\mathrm{Ch}}\mathrm{HP}_0(C_c^{\infty,0}(U))\xrightarrow{\chi_*^\ell}\mathrm{H}_{\ell,c}^{[0]}(U,\mathbb{R})\otimes\mathbb{C}.$$

The algebra $C_c^{\infty,0}(U_0,U_1)$ is a dense subalgebra of $C_0(U_1\setminus U_0)$ stable under holomorphic functional calculus and the inclusion $C_c^{\infty,0}(U_0,U_1)\hookrightarrow C_0(U_1\setminus U_0)$ induces an isomorphism [19]:

$$K_0(C_c^{\infty,0}(U_0,U_1))\cong K_0(C_0(U_1\setminus U_0)).$$

This remark yields for any open pair (U_0,U_1) of V_Ω, to a relative longitudinal Chern character

$$ch_\ell : K_0(C_0(U_1\setminus U_0))\cong K_0(C_c^{\infty,0}(U_0,U_1))\longrightarrow \mathrm{H}_{\ell,c}^{[0]}(U_0,U_1,\mathbb{R}),$$

obtained again from the composite map

$$K_0(C_c^{\infty,0}(U_0,U_1))\xrightarrow{\mathrm{Ch}}\mathrm{HP}_0(C_c^{\infty,0}(U_0,U_1))\xrightarrow{\chi_*^\ell}\mathrm{H}_{\ell,c}^{[0]}(U_0,U_1,\mathbb{R})\otimes\mathbb{C}$$

In the same way, we can define an odd Chern character valued in the odd longitudinal cohomology. We are now in a position to state the compatibity of the longitudinal Chern character with respect to the six term exact sequences associated with a relative open pair (U_0,U_1) of V_Ω.

Theorem 6. *The longitudinal Chern character intertwines the two following exact sequences:*

$$\begin{tikzcd} K_0(C_0^{\infty,0}(U_0)) \arrow[r] & K_0(C_0^{\infty,0}(U_1)) \arrow[r] & K_0(C_0^{\infty,0}(U_0,U_1)) \arrow[d] \\ K_1(C_0^{\infty,0}(U_0,U_1)) \arrow[u] & K_1(C_0^{\infty,0}(U_1)) \arrow[l] & K_1(C_0^{\infty,0}(U_0)) \arrow[l] \end{tikzcd}$$

and

$$\begin{tikzcd} \mathrm{H}_{\ell,c}^{[0]}(U_0,\mathbb{R}) \arrow[r] & \mathrm{H}_{\ell,c}^{[0]}(U_1,\mathbb{R}) \arrow[r] & \mathrm{H}_{\ell,c}^{[0]}(U_0,U_1,\mathbb{R}) \arrow[d] \\ \mathrm{H}_{\ell,c}^{[1]}(U_0,U_1,\mathbb{R}) \arrow[u] & \mathrm{H}_{\ell,c}^{[1]}(U_1,\mathbb{R}) \arrow[l] & \mathrm{H}_{\ell,c}^{[1]}(U_0,\mathbb{R})) \arrow[l] \end{tikzcd}.$$

Proof. Recall that algebras like $C_c^{\infty,0}(U)$ for an open subset U of V_Ω satisfy topological excision [26, 49] and that the Chern-Connes character intertwines both six term exact sequence in K-theory and in periodic cyclic homology except that the boundary map from HP_0 to HP_1 must be multiplied by a factor $2i\pi$, see [49]. The theorem is then a consequence of proposition 5. $\square$

4. Proof of the Bellissard conjecture

4.1. **The measured index theorem.** We shall assume as before that p is even. Let again $\pi : \mathbb{R}^p \to \mathbb{T}^p$ be the covering projection. Denote as before by V_Ω the suspended compact space $V_\Omega = \mathbb{R}^p \times_{\mathbb{Z}^p} \Omega$. Recall that the leaves of the foliation F that we consider on V_Ω, are the quotients of $\{\omega\} \times \mathbb{R}^p$ by the isotropy subgroup for ω, when ω runs over Ω. We shall use in this subsection the measure μ that we have fixed on Ω. This measure is invariant under the action of the group $\mathbb{Z}^p$ by hypothesis.

Let us recall the construction of the Ruelle-Sullivan morphism associated with the $\mathbb{Z}^p$-invariant measure μ on the foliated bundle V_Ω. We denote as before by $\Omega^k_\ell(V_\Omega, \mathbb{R})$ the space of longitudinal k-forms on V_Ω. Using the morphism $\Psi_{\mathbb{Z}^p} : \Omega^p_\ell(V_\Omega, \mathbb{R}) \longrightarrow C(\Omega, \mathbb{R})$ defined in section 2.2 we define:

$$C_{\mu,\mathbb{Z}^p} : \Omega^p_\ell(V_\Omega, \mathbb{R}) \longrightarrow \mathbb{R} \text{ by } C_{\mu,\mathbb{Z}^p}(\phi) := \mu \circ \Psi_{\mathbb{Z}^p} = \left\langle \mu \otimes \int_{\mathbb{R}^p}, \chi\phi \right\rangle.$$

where χ is the characteristic function of the open set $U =]0,1[^p$ in $\mathbb{R}^p$. According to proposition 3 and since μ factorizes through the coinvariants, the map $C_{\mu,\mathbb{Z}^p}$ induces a well defined map $[C_{\mu,\mathbb{Z}^p}] : \mathrm{H}^p_\ell(V_\Omega, \mathbb{R}) \to \mathbb{R}$.

When we consider the Dirac operator along the leaves of V_Ω, twisted by a vector bundle E over V_Ω, the index theorem is not a consequence of the Atiyah covering theorem [2]. Nevertheless, Connes' measured index theorem [20] gives, in our situation, informations on the solutions of longitudinal elliptic equations on the non compact foliated space $\Omega \times \mathbb{R}^p$. Let us state this index formula in our case.

Theorem 7. *Under the above assumptions, the measured index of the longitudinal Dirac operator ∂^e_Ω with coefficients in the vector bundle E, associated with the indempotent e as before, is given by:*

$$\tau^\mu_*(\mathrm{Ind}_{V_\Omega}(\partial^e_\Omega)) = \langle ch_\ell([e]), [C_{\mathbb{Z}^p,\mu}]\rangle.$$

Proof. We apply Connes' measured index theorem in the foliated space V_Ω according to the extended version given in [47][page 261] for foliated spaces. This gives:

$$\tau^\mu_*(\mathrm{Ind}_{V_\Omega}(\partial^e_\Omega)) = \langle \mathrm{Ch}_\ell(E)\hat{A}(F), [C_\mu]\rangle,$$

where $\hat{A}(F)$ is the $\hat{A}$ genus of the longitudinal bundle F of V_Ω (see [47]), $[C_\mu]$ is the Ruelle-Sullivan current associated with the invariant measure μ on V_Ω [47], and Ch_ℓ is the longitudinal Chern character also defined in [47]. Since $V_\Omega \setminus \Omega\times]0,1[^p$ is longitudinally negligible, the pairing of the Ruelle-Sullivan current C_μ with longitudinal differential forms on V_Ω is exactly the pairing of $C_{\mu,\mathbb{Z}^p}$ with the lift of these differential forms to $\Omega \times \mathbb{R}^p$. Now the lift of $\mathrm{Ch}_\ell(E)$ is exactly our longitudinal cohomology class $ch_\ell([e])$. Finally, in the present situation we obviously have $\hat{A}(F) = 1$. ☐

4.2. **An induction formula for mapping tori.** By using the Pimsner-Voiculescu six term exact sequence [53], we obtain for an action of the free abelian group $\mathbb{Z}^p$ on a Cantor set the following short exact sequence

$$0 \longrightarrow \mathrm{Coinv}\, K_i(C(\Omega) \rtimes \mathbb{Z}^{p-1}) \longrightarrow K_{i+1}(C(\Omega) \rtimes \mathbb{Z}^p) \longrightarrow \mathrm{Inv}\, K_{i+1}(C(\Omega) \rtimes \mathbb{Z}^{p-1}) \longrightarrow 0.$$

Here, the action of $\mathbb{Z}^{p-1}$ is obtained by restriction to the $p-1$ first factors of $\mathbb{Z}^p$ and the invariants and the coinvariants are relative to the $\mathbb{Z}-$action through the last factor of $\mathbb{Z}^p$. In this subsection, we state some analogous formulas for the $K-$theory and for the longitudinal cohomology of mapping tori.

Let $\mathbb{Z}^{p-1} \hookrightarrow \mathbb{Z}^p$ and $\mathbb{R}^{p-1} \hookrightarrow \mathbb{R}^p$ be the inclusions corresponding to the $p-1$ first factors. We denote by V'_Ω the mapping torus corresponding to this action of $\mathbb{Z}^{p-1}$, i.e. $V'_\Omega = \dfrac{\Omega \times \mathbb{R}^{p-1}}{\mathbb{Z}^{p-1}}$. Then $V'_\Omega\times]0,1[$ can be viewed as an open subset of V_Ω with $V_\Omega \setminus V'_\Omega\times]0,1[= V'_\Omega$. The first step is to identify $\mathrm{H}^*_{\ell,c}(V'_\Omega\times]0,1[, V_\Omega, \mathbb{R})$ with $\mathrm{H}^*_\ell(V'_\Omega, \mathbb{R})$. Recall that by definition, $\mathrm{H}^*_{\ell,c}(V'_\Omega\times]0,1[, V_\Omega, \mathbb{R})$ is the cohomology of the quotient complex in the exact sequence

$$0 \longrightarrow \Omega^*_{\ell,c}(V'_\Omega\times]0,1[, \mathbb{R}) \longrightarrow \Omega^*_\ell(V_\Omega, \mathbb{R}) \longrightarrow \Omega^*_{\ell,c}(V'_\Omega\times]0,1[, V_\Omega, \mathbb{R}) \longrightarrow 0.$$

If λ is a longitudinal form on V_Ω , then it splits in a unique way into $\lambda = \lambda_1 + \lambda_2 \wedge dt_p$ where λ_1 and λ_2 are valued in $\Omega^*_b(\mathbb{R}^{p-1}) \subset \Omega^*_b(\mathbb{R}^p)$. Then

$$\Omega^*_\ell(V_\Omega, \mathbb{R}) \ni \lambda \longmapsto \lambda_1|_{\Omega\times\mathbb{R}^{p-1}\times\{0\}} \in \Omega^*_\ell(V'_\Omega, \mathbb{R})$$

induces a well defined map

$$\Lambda_1 : \Omega^*_{\ell,c}(V'_\Omega \times]0,1[, V_\Omega, \mathbb{R}) \longrightarrow \Omega^*_\ell(V'_\Omega, \mathbb{R})\,.$$

It is straightforward to check that Λ_1 is a morphism of graded differential algebras.

Lemma 5. *Λ_1 is a quasi-isomorphism, i.e. it induces an isomorphism*

$$\mathrm{H}^*_{\ell,c}(V'_\Omega \times]0,1[, V_\Omega, \mathbb{R}) \cong \mathrm{H}^*_\ell(V'_\Omega, \mathbb{R})\,.$$

Proof. We construct a quasi-inverse for Λ_1 as follows. Let ϕ be a smooth function on $\mathbb{R}$ which is compactly supported in $]-1/2, 1/2[$ and equal to 1 in a neighbourhood of 0. Let λ be a longitudinal form on V'_Ω. We define a longitudinal form $\tilde{\lambda} \in \Omega^*_\ell(V_\Omega, \mathbb{R})$ by setting

$$\tilde{\lambda}(\omega, x_1, x_2, \cdots, x_p) = \phi(x_p)\lambda(\omega, x_1, x_2, \cdots, x_{p-1})$$

if $(\omega, x_1, x_2, \cdots, x_p) \in \Omega \times \mathbb{R}^{p-1} \times]-1/2, 1/2]$ and by requiring $\mathbb{Z}^p-$invariance to extend it to $\Omega \times \mathbb{R}^p$. Since ϕ is constant in a neighbourhood of 0, the equality $d_\ell\tilde{\lambda} = \widetilde{d_\ell\lambda}$ holds modulo $\Omega^*_{\ell,c}(V'_\Omega \times]0,1[, \mathbb{R})$, and therefore

$$\Omega^*_\ell(V'_\Omega, \mathbb{R}) \ni \lambda \longmapsto \tilde{\lambda} \in \Omega^*_\ell(V_\Omega, \mathbb{R})$$

induces a morphism of complexes $\Lambda_2 : \Omega^*_\ell(V'_\Omega, \mathbb{R}) \longrightarrow \Omega^*_{\ell,c}(V'_\Omega \times]0,1[, V_\Omega, \mathbb{R})$. It is clearly a left inverse for Λ_1. To prove that it is a quasi-right-inverse, we use a Poincaré lemma. By using the above splitting $\lambda = \lambda_1 + \lambda_2 \wedge dt_p$ for $\lambda \in \Omega^k_\ell(V_\Omega, \mathbb{R})$, we define $K(\lambda) \in \Omega^{k-1}_\ell(V_\Omega, \mathbb{R})$ by setting

$$K(\lambda)(\omega, x_1, x_2, \cdots, x_p) = (-1)^p \phi(x_p) \int_0^{x_p} \lambda_2(\omega, x_1, x_2, \cdots, x_{p-1}, t_p)dt_p,$$

if $(\omega, x_1, x_2, \cdots, x_{p-1}, x_p) \in \Omega \times \mathbb{R}^{p-1} \times]-1/2, 1/2]$. We extend to $\Omega \times \mathbb{R}^p$ again by requiring $\mathbb{Z}^p-$invariance. The subcomplex $\Omega^*_{\ell,c}(V'_\Omega \times]0,1[, \mathbb{R}) \subset \Omega^*_\ell(V_\Omega, \mathbb{R})$ is stable under K and hence, K induces a degree -1 endomorphism of the complex $\Omega^*_{\ell,c}(V'_\Omega \times]0,1[, V_\Omega, \mathbb{R})$. Now, it is straightforward to check that

$$K \circ d_\ell + d_\ell \circ K = \mathrm{Id} - \Lambda_2 \circ \Lambda_1\,.$$

□

The inclusion

$$(\omega, x_1, x_2, \cdots, x_{p-1}) \mapsto (\omega, x_1, x_2, \cdots, x_{p-1}, 0)$$

induces an isomorphism $K_*(C^{\infty,0}_c(V'_\Omega \times]0,1[, V_\Omega)) \cong K_*(C^{\infty,0}(V'_\Omega))$ and the longitudinal Chern character intertwines this isomorphism with the isomorphism of Lemma 5. More precisely:

Lemma 6. *The following diagram is commutative*

$$\begin{tikzcd} K_*(C^{\infty,0}_c(V'_\Omega \times]0,1[, V_\Omega)) \arrow[r] \arrow[d] & K_*(C^{\infty,0}(V'_\Omega)) \arrow[d] \\ \mathrm{H}^*_{\ell,c}(V'_\Omega \times]0,1[, V_\Omega, \mathbb{R}) \arrow[r, "\Lambda_1"] & \mathrm{H}^*_\ell(V'_\Omega, \mathbb{R}) \end{tikzcd}$$

where the vertical arrows are both longitudinal Chern characters.

Proof. This is done by using naturality of the Chern character in cyclic periodic homology and by observing that the following diagram is commutative:

$$\begin{tikzcd} \mathcal{C}_*(C^{\infty,0}_c(V'_\Omega \times]0,1[, V_\Omega)) \arrow[r] \arrow[d, "\chi^\ell_*"'] & \mathcal{C}_*(C^{\infty,0}(V'_\Omega)) \arrow[d, "\chi^\ell_*"] \\ \mathrm{H}^*_{\ell,c}(V'_\Omega \times]0,1[, V_\Omega, \mathbb{R}) \arrow[r, "\Lambda_1"] & \mathrm{H}^*_\ell(V'_\Omega, \mathbb{R}) \end{tikzcd}$$

Here the horizontal arrows are induced by the inclusion

$$(\omega, x_1, x_2, \cdots, x_{p-1}) \longmapsto (\omega, x_1, x_2, \cdots, x_{p-1}, 0)$$

and the vertical arrows are the HKR maps defined in Section 3.2. □

We now investigate the behaviour of the longitudinal Chern character with respect to Bott periodicity. To this end, we define a morphism $\mathrm{H}^i_\ell(V'_\Omega, \mathbb{R}) \to \mathrm{H}^{i+1}_{\ell,c}(V'_\Omega \times]0,1[, \mathbb{R})$ in the following way. Fix a smooth compactly supported function $\phi :]0,1[\to \mathbb{R}^+$ such that $\int_{]0,1[} \phi = 1$. For any longitudinal form λ in $\Omega^i_\ell(V'_\Omega)$, we define

$$\hat{\lambda}(\omega, x_1, x_2, \cdots, x_p) = \phi(x_p)\lambda(\omega, x_1, x_2, \cdots, x_{p-1}) \wedge dx_p,$$

if $(\omega, x_1, x_2, \cdots, x_p) \in \Omega \times \mathbb{R}^{p-1} \times]0,1[$ and extend it to $\Omega \times \mathbb{R}^p$ by requiring $\mathbb{Z}^p-$invariance. This is obviously a morphism of complexes and induces a morphism $\Lambda : \mathrm{H}^i_\ell(V'_\Omega, \mathbb{R}) \to \mathrm{H}^{i+1}_{\ell,c}(V'_\Omega \times]0,1[, \mathbb{R})$.

Lemma 7.
- $\Lambda : \mathrm{H}^i_\ell(V'_\Omega, \mathbb{R}) \longrightarrow \mathrm{H}^{i+1}_{\ell,c}(V'_\Omega \times]0,1[, \mathbb{R})$ *is a isomorphism.*
- *The following diagram is commutative*

$$\begin{array}{ccc} K_i(C^{\infty,0}(V'_\Omega)) & \longrightarrow & K_{i+1}(C^{\infty,0}_c(V'_\Omega \times]0,1[)) \\ \downarrow & & \downarrow \\ \mathrm{H}^{[i]}_\ell(V'_\Omega, \mathbb{R}) & \xrightarrow{\Lambda} & \mathrm{H}^{[i+1]}_{\ell,c}(V'_\Omega \times]0,1[, \mathbb{R}) \end{array}$$

where the top horizontal arrow is the Bott periodicity isomorphism.

Proof.
Every longitudinal form λ of $\Omega^*_{\ell,c}(V'_\Omega \times]0,1[, \mathbb{R})$ splits in a unique way into $\lambda = \lambda_1 + \lambda_2 \wedge dt_p$ where λ_1 and λ_2 are valued in the space of exterior forms on $\mathbb{R}^{p-1}$ (viewed as a subspace of the space of exterior forms on $\mathbb{R}^p$). Moreover,

$$\Omega^*_{\ell,c}(V'_\Omega \times]0,1[, \mathbb{R}) \ni \lambda \longmapsto \int_{]0,1[} \lambda_2(\,\cdot\,, x_p)dx_p \in \Omega^*_\ell(V'_\Omega, \mathbb{R})$$

is a morphism of complexes which induces a morphism

$$\Lambda' : \mathrm{H}^{[i+1]}_{\ell,c}(V'_\Omega \times]0,1[, \mathbb{R}) \longrightarrow \mathrm{H}^{[i]}_\ell(V'_\Omega, \mathbb{R}).$$

This is obviously a left inverse for Λ. If $\lambda = \lambda_1 + \lambda_2 \wedge dt_p$ is the above splitting for $\lambda \in \Omega^k_{\ell,c}(V'_\Omega \times]0,1[, \mathbb{R})$, then we define the longitudinal form $K(\lambda)$ of $\Omega^{k-1}_{\ell,c}(V'_\Omega \times]0,1[, \mathbb{R})$ by setting

$$K(\lambda)(\omega, t_1, \cdots, t_p) = (-1)^{p-1}\left(\int_0^{t_p} \lambda_2(\omega, t_1, \cdots, t_{p-1}, x_p)dx_p \right.$$
$$\left. - \int_0^{t_p} \phi(x_p)dx_p \int_0^1 \lambda_2(\omega, t_1, \cdots, t_{p-1}, x_p)dx_p\right).$$

if $(\omega, x_1, x_2, \cdots, x_p) \in \Omega \times \mathbb{R}^{p-1} \times]0,1[$ and extend it to $\Omega \times \mathbb{R}^p$ by requiring $\mathbb{Z}^p-$invariance. One can check that $d_\ell \circ K + K \circ d_\ell = \mathrm{Id} - \Lambda \circ \Lambda'$ and thus, that Λ' is a quasi inverse for Λ. This prove the first item of the lemma.

To prove the second item, we identify Λ with the boundary map of the extension that provides the Bott isomorphism. Let $\Omega^k_{b,c}(\mathbb{R}^{p-1} \times]0,1])$ be the space of k-differential forms ϕ on $\mathbb{R}^{p-1} \times]0,1]$, compactly supported on the factor $]0,1]$, with bounded derivatives at all orders and such that for some λ in $]0,1[$ the restriction of ϕ to $\mathbb{R}^{p-1} \times]\lambda, 1]$ does not depend on the second factor. We define the algebra of longitudinally smooth functions on $V'_\Omega \times]0,1]$

$A = \{\phi : \Omega \longrightarrow \Omega^0_{b,c}(\mathbb{R}^{p-1} \times]0,1]) \otimes \mathbb{C}$ continuous and $\mathbb{Z}^{p-1}$-equivariant (the $\mathbb{Z}^{p-1}$-action is on the $\mathbb{R}^{p-1}$-factor)$\}$.

We also define the longitudinal k-forms on $V'_\Omega\times]0,1]$

$$\Omega^k(A,\mathbb{R})=\{\phi:\Omega\longrightarrow\Omega^k_{b,c}(\mathbb{R}^{p-1}\times]0,1])\text{ continuous and }\mathbb{Z}^{p-1}\text{-equivariant}\}.$$

Then, $\Omega^*(A,\mathbb{R})$ is equipped with a differential $\Omega^k(A,\mathbb{R})\to\Omega^{k+1}(A,\mathbb{R})$ such that for $\phi\in\Omega_k(A,\mathbb{R})$, $d_A(\phi)$ is the $\mathbb{Z}^{p-1}$-equivariant map $\omega\mapsto d\phi(\omega,\cdot)$.

The boundary of the short exact sequence

$$0\longrightarrow\Omega^*_{\ell,c}(V'_\Omega\times]0,1[,\mathbb{R})\longrightarrow\Omega^*(A,\mathbb{R})\longrightarrow\Omega^*_\ell(V'_\Omega,\mathbb{R})\longrightarrow 0$$

induces an morphism $\mathrm{H}^{i+1}_{\ell,c}(V'_\Omega\times]0,1[,\mathbb{R})\longrightarrow\mathrm{H}^i_\ell(V'_\Omega,\mathbb{R})$ and a straightforward computation shows that it coincides with Λ.

Recall that Bott periodicity in K-theory and in periodic cyclic homology are both given by the boundary map of the short exact sequence

$$0\longrightarrow C^{\infty,0}_c(V'_\Omega\times]0,1[)\longrightarrow A\longrightarrow C^{\infty,0}_c(V'_\Omega)\longrightarrow 0,$$

and thus according to theorem 4, it is enough to check that the following diagram is commutative

$$\begin{tikzcd}
\mathrm{HP}_{k+1}(C^{\infty,0}_c(V'_\Omega\times]0,1[)) \arrow[r, "\chi^\ell_{k+1}"] & \mathrm{H}^{[k+1]}_{\ell,c}(V'_\Omega\times]0,1[,\mathbb{R})\otimes\mathbb{C} \\
\mathrm{HP}_k(C^{\infty,0}(V'_\Omega)) \arrow[u] \arrow[r, "\chi^\ell_k"] & \mathrm{H}^{[k]}_\ell(V'_\Omega,\mathbb{R})\otimes\mathbb{C} \arrow[u, "\Lambda"']
\end{tikzcd}$$

where the left vertical map is the Bott map and χ^ℓ_* are the HKR homomorphisms defined in 3.2. We have a HKR map

$$\chi^A_n:\mathcal{C}_n(A)\longrightarrow\Omega^n(A,\mathbb{R})\otimes\mathbb{C}$$

such that for every $f_0,\cdots,f_n$ in A,

$$\chi^A_n(f^0\hat{\otimes}\cdots\hat{\otimes}f^n)=\left(\frac{1}{2i\pi}\right)^{\left[\frac{n+1}{2}\right]}\frac{1}{n!}f^0d_Af^1\wedge\cdots\wedge d_Af^n.$$

By the same argument of proposition 4, this is a chain map (up to rescaling the odd differential) beetween $(\mathcal{C}_*(A),B+b)$ and $(\Omega^*(A,\mathbb{R})\otimes\mathbb{C},d_A)$ and the following diagram commutes

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{C}_n(C^\infty_c(V'_\Omega\times]0,1[)) \arrow[r] \arrow[d, "\chi^\ell_*"] & \mathcal{C}_n(A) \arrow[r] \arrow[d, "\chi^A_*"] & \mathcal{C}_n(C^\infty(V'_\Omega)) \arrow[r] \arrow[d, "\chi^\ell_*"] & 0 \\
0 \arrow[r] & \Omega^*_{\ell,c}(V'_\Omega\times]0,1[,\mathbb{R})\otimes\mathbb{C} \arrow[r] & \Omega_*(A,\mathbb{R})\otimes\mathbb{C} \arrow[r] & \Omega^*_\ell(V'_\Omega,\mathbb{R})\otimes\mathbb{C} \arrow[r] & 0
\end{tikzcd}$$

By naturality of the boundaries, the (HKR) maps intertwine the boundaries associated with the above two exact sequences, and thus we get the result. ☐

Remark 4. With the notations of Section 2.2, the composite map

$$\mathrm{H}^{p-1}_\ell(V'_\Omega,\mathbb{R})\xrightarrow{\Lambda}\mathrm{H}^p_{\ell,c}(V'_\Omega\times]0,1[,\mathbb{R})\hookrightarrow\mathrm{H}^p_\ell(V_\Omega,\mathbb{R})\xrightarrow{\Psi_{\mathbb{Z}^p}}C(\Omega)_{\mathbb{Z}^p}$$

is equal to $\Psi_{\mathbb{Z}^{p-1}}$.

There is an action of $\mathbb{Z}$ on V'_Ω through the inclusion $\mathbb{Z}\hookrightarrow\mathbb{Z}^p$ of the last factor. This induces an action of $\mathbb{Z}$ on $K_i(C(V'_\Omega))$ and on $\mathrm{H}^{[i]}_\ell(V'_\Omega,\mathbb{R})$. The longitudinal Chern character is then a $\mathbb{Z}-$equivariant morphism. We denote the invariants of this $\mathbb{Z}-$action by Inv $K_i(C(V'_\Omega))$ and Inv $\mathrm{H}^{[i]}_\ell(V'_\Omega,\mathbb{R})$, while the coinvariants are denoted by Coinv $K_i(C(V'_\Omega))$ and Coinv $\mathrm{H}^{[i]}_\ell(V'_\Omega,\mathbb{R})$.

Theorem 8. *There is a commutative diagram*

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Coinv}\ K_i(C(V'_\Omega)) \arrow[r] \arrow[d, "ch_\ell"] & K_{i+1}(C(V_\Omega)) \arrow[r] \arrow[d, "ch_\ell"] & \mathrm{Inv}\ K_{i+1}(C(V'_\Omega)) \arrow[r] \arrow[d, "ch_\ell"] & 0 \\
0 \arrow[r] & \mathrm{Coinv}\ \mathrm{H}_\ell^{[i]}(V'_\Omega, \mathbb{R}) \arrow[r] & \mathrm{H}_\ell^{[i+1]}(V_\Omega, \mathbb{R}) \arrow[r] & \mathrm{Inv}\ \mathrm{H}_\ell^{[i+1]}(V'_\Omega, \mathbb{R}) \arrow[r] & 0
\end{tikzcd}$$

where the bottom epimorphism preserves the degree, the bottom inclusion is induced by the map

$$\Omega^i_\ell(V'_\Omega, \mathbb{R}) \ni \lambda \mapsto \hat{\lambda} \in \Omega^{i+1}_{\ell,c}(V'_\Omega \times]0,1[, \mathbb{R}) \subset \Omega^{i+1}_\ell(V_\Omega, \mathbb{R})$$

and where the top inclusion is induced by the composite map

$$K_i(C(V'_\Omega) \cong K_{i+1}(C(V'_\Omega \times]0,1[)) \longrightarrow K_{i+1}(C(V_\Omega)).$$

Proof. By using the six-term exact sequence associated with the relative open pair $(V'_\Omega \times]0,1[, V_\Omega)$, Theorem 6, and Lemmas 6 and 7, we deduce two six-term exact sequences

$$\begin{tikzcd}
K_1(C^{\infty,0}(V'_\Omega)) \arrow[r] & K_0(C^{\infty,0}(V_\Omega)) \arrow[r] & K_0(C^{\infty,0}(V'_\Omega)) \arrow[d] \\
K_1(C^{\infty,0}(V'_\Omega) \arrow[u] & K_1(C^{\infty,0}(V_\Omega)) \arrow[l] & K_0(C^{\infty,0}(V'_\Omega)) \arrow[l]
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathrm{H}^o_\ell(V'_\Omega, \mathbb{R}) \arrow[r] & \mathrm{H}^e_\ell(V_\Omega, \mathbb{R}) \arrow[r] & \mathrm{H}^e_\ell(V'_\Omega, \mathbb{R}) \arrow[d] \\
\mathrm{H}^o_\ell(V'_\Omega, \mathbb{R}) \arrow[u] & \mathrm{H}^o_\ell(V_\Omega, \mathbb{R}) \arrow[l] & \mathrm{H}^e_\ell(V'_\Omega, \mathbb{R}) \arrow[l]
\end{tikzcd}$$

intertwined by the longitudinal Chern characters. The computation of the boundary maps is standard. If we denote by e^p the p^{th} generator of $\mathbb{Z}^p$, they are both given by the identity morphism minus the morphism induced by the action of e^p on ${V_\Omega}'$. Therefore, the kernel and the cokernel of the first boundary map are respectivly Inv $K_{i+1}(C(V'_\Omega))$ and Coinv $K_i(C(V'_\Omega))$, and the kernel and the cokernel of the second boundary map are respectivly Inv $\mathrm{H}_\ell^{[i+1]}(V_\Omega, \mathbb{R})$ and Coinv $\mathrm{H}_\ell^{[i]}(V'_\Omega, \mathbb{R})$. This complete the proof. ☐

4.3. **Integrality of the Chern character.** Recall that there exists a Chern character in Čech cohomology $ch : K^*(X) \to \mathrm{H}^*(X, \mathbb{R})$ for every topological space X. An analougous of theorem 8 can be stated with ch and with the Čech cohomology. Notice that using this induction formula for the Čech cohomology (with integral or rational coefficients) of the mapping torus, we can show that the mapping torus V_Ω has cohomological dimension p.

Proposition 6. [47] *The composition* $K^*(V_\Omega) \overset{ch}{\to} \mathrm{H}^*(V_\Omega, \mathbb{R}) \to \mathrm{H}^*_\ell(V_\Omega, \mathbb{R})$*, where the second arrow is the morphism of Remark 1, coincides with* $ch_\ell : K^*(V_\Omega) \to \mathrm{H}^*_\ell(V_\Omega, \mathbb{R})$.

The range of ch has been studied in [30] and the following theorem is stated.

Theorem 9. [30]

- *The morphism* $ch : K_*(V_\Omega) \to \mathrm{H}^*(V_\Omega, \mathbb{R})$ *is injective with homogeneous image (i.e. Its image is graded by the grading of* $\mathrm{H}^*(V_\Omega, \mathbb{R})$ *).*
- *Moreover, the image of* $ch : K_*(V_\Omega) \to \mathrm{H}^*(V_\Omega, \mathbb{R})$ *is isomorphic to* $\oplus_j \mathrm{H}_j(\mathbb{Z}^p, C(\Omega, \mathbb{Z}))$.

It was pointed out to us by the referee that the proof [30] possibly contains a mistake and a complement to this computation has been recently announced in [35].

Proposition 7. *Let us denote by ch_ℓ^k the $k-$component of the longitudinal Chern character lying in $\mathrm{H}_\ell^k(V_\Omega,\mathbb{R})$. If $i\in\{0,1\}$ is equal to p modulo 2, then $\psi_{\mathbb{Z}^p}(ch_\ell^p(K^i(V_\Omega))$ belongs to $C(\Omega,\mathbb{Z})_{\mathbb{Z}^p}$.*

Proof. We proceed inductively. This is clearly true in low degree. Let x be an element of $K_0(C(V_\Omega))$. According to Theorem 9, there is an element $x'\in K_0(C(V_\Omega))$ such that $ch(x')=ch^p(x)$ and hence $ch_\ell(x')=ch_\ell^p(x)$ by Proposition 6. Recall from theorem 8 the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Coinv}\ K_{i-1}(C(V'_\Omega)) \arrow[r] \arrow[d, "ch_\ell"] & K_i(C(V_\Omega)) \arrow[r] \arrow[d, "ch_\ell"] & \mathrm{Inv}\ K_i(C(V'_\Omega)) \arrow[r] \arrow[d, "ch_\ell"] & 0 \\
0 \arrow[r] & \mathrm{Coinv}\ \mathrm{H}_\ell^{[i-1]}(V'_\Omega,\mathbb{R}) \arrow[r] & \mathrm{H}_\ell^{[i]}(V_\Omega,\mathbb{R}) \arrow[r] & \mathrm{Inv}\ \mathrm{H}_\ell^{[i]}(V'_\Omega,\mathbb{R}) \arrow[r] & 0
\end{tikzcd}$$

Since $ch: K_*(C(V'_\Omega))\to \mathrm{H}^*(V'_\Omega,\mathbb{R})$ is injective and since $\mathrm{H}^p(V'_\Omega,\mathbb{R})=0$, the top epimorphism maps x' to 0 and thus x' comes from an element of $K_{i-1}(C(V'_\Omega))$ with longitudinal Chern character of degree $p-1$. The proof is completed by using remark 4. □

We point out that this integrality property (Proposition 7) is also implicitely used in [7] and in [36]. Notice that in dimension 3, since the Chern character can be valued in integral cohomology and is an isomorphism, the proof of proposition 7 does not rely on the result of [30].

We are now in position to prove the Bellissard conjecture.

Theorem 10. *Let $(\Omega,\mathbb{Z}^p)$ be a dynamical system with Ω a Cantor set and let μ be a $\mathbb{Z}^p-$invariant measure on Ω. Assume that Ω has no non-trivial invariant compact-open subset. Let τ_*^μ be the additive map induced by the trace τ^μ, associated with μ, on the K-theory of the C^*-algebra $C(\Omega)\rtimes\mathbb{Z}^p$. Then we have:*

$$\tau_*^\mu(K_0(C(\Omega)\rtimes\mathbb{Z}^p))=\mathbb{Z}[\mu].$$

Proof. By Lemma 2, we can assume that p is even. From Theorem 1 we deduce that:

$$\tau_*^\mu(K_0(C(\Omega)\rtimes\mathbb{Z}^p))=\{\tau_*^\mu(\mathrm{Ind}_{V_\Omega}(\partial_\Omega^e))-\tau_*^\mu(\mathrm{Ind}_{V_\Omega}(\partial_\Omega^{e'})),[e]-[e']\in K_0(C^{\infty,0}(V_\Omega))\}.$$

Using Theorem 7, we obtain:

$$\tau_*^\mu(K_0(C(\Omega)\rtimes\mathbb{Z}^p))=\langle ch_\ell^p(K^0(V_\Omega)),[C_{\mu,\mathbb{Z}^p}]\rangle.$$

By definition of $C_{\mu,\mathbb{Z}^p}$, we have:

$$\langle x,[C_{\mu,\mathbb{Z}^p}]\rangle=\langle\psi_{\mathbb{Z}^p}(x),\mu\rangle,\quad \forall x\in \mathrm{H}_\ell^p(V_\Omega,\mathbb{R}).$$

Therefore,

$$\langle ch_\ell^p(e),[C_{\mu,\mathbb{Z}^p}]\rangle=\langle\psi_{\mathbb{Z}^p}(ch_\ell^p(e)),\mu\rangle.$$

But according to Proposition 7, we have:

$$\psi_{\mathbb{Z}^p}(ch_\ell^p(e))\in C(\Omega,\mathbb{Z})_{\mathbb{Z}^p}.$$

Hence:

$$\tau_*^\mu(K_0(C(\Omega)\rtimes\mathbb{Z}^p))\subset\mu(C(\Omega,\mathbb{Z})_{\mathbb{Z}^p})=\mathbb{Z}[\mu].$$

Since the opposite inclusion is checked in Lemma 1, the proof is complete. □

Remark 5. The homological computations carried out in the present paper actually show that the longitudinal (HKR) map χ_*^ℓ together with the map ρ_* defined in Section 1 are isomorphisms and are induced by the corresponding maps at the E^1 level of the corresponding spectral sequences.

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